MIDTERM TEST SAMPLE SOLUTION

1. In a weighted graph with start node $s$, there are often multiple shortest paths from $s$ to any other node. We want to use Dijkstra’s algorithm to count them (so assume no negative edge weights). To each node $v$, we add a field $v.numPaths$ initialized to 0 (with $s.numPaths$ initialized to 1). Modify the RELAX routine so that Dijkstra’s algorithm will determine both the length of the shortest path to all other nodes and the number of such shortest paths.

(sol’n:)

procedure relax(u,v)
    if ( v.d > u.d + W(u,v) )
        then v.d = u.d + W(u,v)
            v.numPaths = u.numPaths
    else if (v.d == u.d + W(u,v) )
        then v.numPaths = v.numPaths + u.numPaths

□

2. Using the graph of figure 1, show the start and finish times determined by DFS. Using those finish times, provide a topological sort of the nodes of the graph. As on the homework, visit nodes in alphabetical order.

(sol’n:)

The annotated graph shown in figure 1 shows the start/finish times from a DFS. Listing the nodes in reverse order of finish time gives the requested topological sort: $d c f b g a e$ □

3. Illustrate the Floyd-Warshall algorithm on the graph with the following weight matrix.

\[
\begin{pmatrix}
0 & 3 & \infty & 2 \\
3 & 0 & 8 & 6 \\
\infty & 8 & 0 & 5 \\
2 & 6 & 5 & 0
\end{pmatrix}
\]

Show the intermediate matrices $D^{(k)}$ for $k = 1, 2, 3, 4$. Note that the graph is undirected (and hence the matrix is symmetric). You do not need to show how each entry was computed.

(sol’n:)

The input matrix above is the initial $D^{(0)}$. The remaining computed matrices are below (boxed numbers indicate a change from the previous matrix).

\[
D^{(1)} = \begin{pmatrix}
0 & 3 & \infty & 2 \\
3 & 0 & 8 & \boxed{5} \\
\infty & 8 & 0 & 5 \\
2 & \boxed{5} & 5 & 0
\end{pmatrix}
\]
\[
D^{(2)} = \begin{pmatrix}
0 & 3 & 11 & 2 \\
3 & 0 & 8 & 5 \\
11 & 8 & 0 & 5 \\
2 & 5 & 5 & 0
\end{pmatrix} = D^{(3)}
\]

\[
D^{(4)} = \begin{pmatrix}
0 & 3 & 7 & 2 \\
3 & 0 & 8 & 5 \\
7 & 8 & 0 & 5 \\
2 & 5 & 5 & 0
\end{pmatrix}
\]

4. Imagine we are on a tour of some popular tourist spots and we wish to visit each one of them. They are all along a single road at mileposts \(m_0, m_1, m_2, \ldots, m_n\), and we will stop at each one in that order. The choice we have to make is which taxi company to use to take us from one location to the next. The start point is \(m_0\).

The choices are taxi company S (slow) and company T (tedious). Company S charges \(s_i\) dollars per mile to travel from location \(i\) to \(i+1\), so the charge for that segment is \((m_{i+1} - m_i) \cdot s_i\) dollars. Company T charges a flat rate of \(t\) dollars per segment but if chosen they must be used for three consecutive segments. For example, we could start with company T at location \(i\), have them take us to \(i+1, i+2\), and finally finish with them at location \(i+3\) for a total of \(3t\) dollars (of course, we could decide to use them for the next three segments). Our goal is to find the cheapest cost for all segments taking us from location 0 to location \(n\).

For example, suppose the mileposts are at locations \((0, 25, 50, 100, 150, 200)\), the costs for company S are \((2, 3, 2, 1, 2)\), and the flat rate for company T is 40. If the travel plan is \((S, T, T, T, S)\) the travel cost is \(25 \cdot 2 + 40 + 40 + 40 + 50 \cdot 2 = 270\) dollars. However the plan \((S, S, T, T, T)\) incurs a cost of \(25 \cdot 2 + 25 \cdot 3 + 40 + 40 + 40 = 245\) dollars.

Define the subproblem \(C(i)\) to be the minimum cost of a plan that starts at location 0 and ends at location \(i\), and so that either (a) company S brought us from \(m_{i-1}\) to \(m_i\) (at a cost of \(s_{i-1}\) per mile, or (b) company T was used on the last three segments (from \(m_{i-3}\) to \(m_{i-2}\) to \(m_{i-1}\) to \(m_i\)). Note that case (b) is only possible if \(i \geq 3\).

- Give the base case or cases for \(C\).
- Provide a recurrence relation for \(C\). (No code needed.)

(sol’n:)

Base cases and recurrence below:

\[
C(i) = \begin{cases}
0 & \text{if } i = 0 \\
(m_i - m_{i-1}) \cdot s_{i-1} + C(i - 1) & \text{if } i = 1, 2 \text{ (can only use company S)} \\
\min[(m_i - m_{i-1}) \cdot s_{i-1} + C(i - 1), 3 \cdot t + C(i - 3)] & \text{if } i \geq 3 \text{ (min of choices (a) and (b) above)}
\end{cases}
\]

The minimum cost to go from location 0 (at \(m_0\)) to location \(n\) (at \(m_n\)) is thus \(C(n)\).
Figure 1: solution for question 2, showing start/finish times