1. Consider the graph below. You will be building a MST for this graph in two ways. When there is a tie on the edge weights, consider the edges or nodes in alphabetical order. (For an edge, this means (a,f) before (b,c) and (c,d) before (c,g).) Indicate in some way the order in which the edges are added to the MST.

(a) Use Kruskal’s method.
(b) Use Prim’s method.

![Graph Image](image-url)  

**Figure 1: for question 1**

[8 points]

2. Modify Dijkstra’s algorithm to count the number of shortest paths from the start node to each other node. It will still need to determine the length of the shortest path from the start node to each other node as well. [5 points]

3. We are given a graph $G = (V, E)$ where $V$ represents a set of locations and $E$ represents a communications channel between two points. We are also given locations $s, t \in V$, and a
reliability function \( r : V \times V \rightarrow [0, 1] \). You need to give an efficient algorithm which will output the reliability of the most reliable path from \( s \) to \( t \) in \( G \). (It is enough to modify Dijkstra’s algorithm.)

For any points \( u, v \in V \), \( r(u, v) \) is the probability that the communication link \((u, v)\) will not fail: \( 0 \leq r(u, v) \leq 1 \). Note that if there is a path with two edges, for example, from \( u \) to \( v \) to \( w \), then the reliability of that path is \( r(u, v) \cdot r(v, w) \). [6 points]

4. Sometimes the shortest path between two points don’t involve many edges. So given a weighted, directed graph \( G = (V, E) \) with no negative-weight cycles, let \( m \) be the maximum over all vertices \( v \in V \) of the minimum number of edges in a shortest path from the source \( s \) to \( v \). (Here, the shortest path is by weight, not the number of edges.) Suggest a simple change to the Bellman-Ford algorithm that allows it to terminate in \( m + 1 \) passes, even if \( m \) is not known in advance. [5 points]

5. (NEW: this problem is now extra credit) Suppose we are given a directed graph \( G \) with \( n \) vertices, and let \( M \) be the \( n \times n \) adjacency matrix corresponding to \( G \).

[Part 1]
Let the product of \( M \) with itself (\( M^2 \)) be defined, for \( 1 \leq i, j \leq n \), as follows:

\[
M^2(i, j) = (M(i, 1) \odot M(1, j)) \oplus \cdots \oplus (M(i, n) \odot M(n, j)),
\]

where \( \oplus \) is the logical or operator and \( \odot \) is the logical and operator. Furthermore, define \( M^3 \) as

\[
M^3(i, j) = (M^2(i, 1) \odot M(1, j)) \oplus \cdots \oplus (M^2(i, n) \odot M(n, j)).
\]

Given these definitions, what do \( M^2(i, j) = 1 \) and \( M^3(i, j) = 1 \) imply about the vertices \( i \) and \( j \)?

[Part 2]
Now suppose that \( G \) is weighted and assume the following

- for \( 1 \leq i \leq n \), \( M(i, i) = 0 \)
- for \( 1 \leq i, j \leq n \), \( M(i, j) = \text{weight}(i, j) \) if \((i, j) \in E\)
- for \( 1 \leq i, j \leq n \), \( M(i, j) = \infty \) if \((i, j) \notin E\)

Similarly let \( M^2(i, j) \) and \( M^3(i, j) \) be defined, for \( 1 \leq i, j \leq n \), as follows:

\[
M^2(i, j) = \min[M(i, 1) + M(1, j), M(i, 2) + M(2, j), \ldots, M(i, n) + M(n, j)] \text{ and}
\]

\[
M^3(i, j) = \min[M^2(i, 1) + M(1, j), M^2(i, 2) + M(2, j), \ldots, M^2(i, n) + M(n, j)].
\]

If \( M^2(i, j) = r \) or \( M^3(i, j) = s \), what may we conclude about the relationship between vertices \( i \) and \( j \)?

Justify your answers. [8 points]

Total: 24 points