Gale-Shapley
example

• **n hospitals, n students**
• one student per hospital
• each hospital ranks all students
• each student ranks all hospitals
unstable

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A–Y is an unstable pair for matching $M = \{ A-Z, B-Y, C-X \}$
the algorithm:

**Gale–Shapley** *(preference lists for hospitals and students)*

**Initialize** $M$ to empty matching.

**While** (some hospital $h$ is unmatched and hasn’t proposed to every student)

$s \leftarrow$ first student on $h$’s list to whom $h$ has not yet proposed.

**If** ($s$ is unmatched)

Add $h$–$s$ to matching $M$.

**Else If** ($s$ prefers $h$ to current partner $h'$)

Replace $h'$–$s$ with $h$–$s$ in matching $M$.

**Else**

$s$ rejects $h$. 
steps

1. s remains matched from the first time they receive a proposal; the sequence of h they are matched to gets better over time
2. the sequence of s that an h proposes to gets worse over time
3. the GS algorithm terminates after at most $n^2$ iterations
4. if h is unmatched at some point in the execution, there is an s to which h has not yet proposed
5. the set M returned at termination is a perfect matching
6. the set M returned is a stable matching (pf on next page)
Claim. In Gale–Shapley matching $M^*$, there are no unstable pairs.

Pf. Consider any pair $h–s$ that is not in $M^*$.

- Case 1: $h$ never proposed to $s$.
  $\Rightarrow$ $h$ prefers its Gale–Shapley partner $s'$ to $s$.
  $\Rightarrow$ $h–s$ is not unstable.

- Case 2: $h$ proposed to $s$.
  $\Rightarrow$ $s$ rejected $h$ (either right away or later)
  $\Rightarrow$ $s$ prefers Gale–Shapley partner $h'$ to $h$.
  $\Rightarrow$ $h–s$ is not unstable.

- In either case, the pair $h–s$ is not unstable. $\blacksquare$