CIS 471/571 (Winter 2020): Introduction to Artificial Intelligence

Lecture 17: Hidden Markov Model

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Source: http://ai.berkeley.edu/home.html
Reminder

- Homework 4: Bayes Nets, HMMs
  - Deadline: March 06th, 2020
Hidden Markov Model
Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring

- Need to introduce time (or space) into our models
Markov Models

- Value of X at a given time is called the state

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action
Conditional Independence

- Basic conditional independence:
  - Past and future independent given the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property

- Note that the chain is just a (growable) BN
  - We can always use generic BN reasoning on it if we truncate the chain at a fixed length
Example Markov Chain: Weather

- States: $X = \{\text{rain, sun}\}$
- Initial distribution: 1.0 sun
- CPT $P(X_t | X_{t-1})$:

| $X_{t-1}$ | $X_t$    | $P(X_t | X_{t-1})$ |
|-----------|----------|-------------------|
| sun       | sun      | 0.9               |
| sun       | rain     | 0.1               |
| rain      | sun      | 0.3               |
| rain      | rain     | 0.7               |
Example Markov Chain: Weather

- Initial distribution: 1.0 sun

- What is the probability distribution after one step?

\[
P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})
\]

\[
0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9
\]
Mini-Forward Algorithm

- Question: What’s P(X) on some day t?

\[ P(x_1) = \text{known} \]

\[ P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t) \]

\[ = \sum_{x_{t-1}} P(x_t | x_{t-1})P(x_{t-1}) \]

Forward simulation
Example Run of Mini-Forward Algorithm

- From initial observation of sun
  \[
  \begin{align*}
  &\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix} & \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} & \begin{pmatrix} 0.84 \\ 0.16 \end{pmatrix} & \begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix} \\
  &P(X_1) & P(X_2) & P(X_3) & P(X_4)
  \end{align*}
  \]
  \rightarrow
  \[
  \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}
  \]
  P(X_\infty)

- From initial observation of rain
  \[
  \begin{align*}
  &\begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} & \begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix} & \begin{pmatrix} 0.588 \\ 0.412 \end{pmatrix} \\
  &P(X_1) & P(X_2) & P(X_3) & P(X_4)
  \end{align*}
  \]
  \rightarrow
  \[
  \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}
  \]
  P(X_\infty)

- From yet another initial distribution \( P(X_1) \):
  \[
  \begin{align*}
  &\begin{pmatrix} p \\ 1-p \end{pmatrix} \\
  &P(X_1)
  \end{align*}
  \]
  \rightarrow
  \[
  \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}
  \]
  P(X_\infty)
Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

- Stationary distribution:
  - The distribution we end up with is called the stationary distribution $P_\infty$ of the chain
  - It satisfies
    \[
    P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)
    \]
Example: Stationary Distributions

**Question:** What’s $P(X)$ at time $t = \infty$?

- $P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$

- $P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$

| $X_{t-1}$ | $X_t$ | $P(X_t|X_{t-1})$ |
|-----------|-------|------------------|
| sun       | sun   | 0.9              |
| sun       | rain  | 0.1              |
| rain      | sun   | 0.3              |
| rain      | rain  | 0.7              |

Also:

- $P_\infty(sun) + P_\infty(rain) = 1$

- $P_\infty(sun) = 3/4$

- $P_\infty(rain) = 1/4$
**Application of Stationary Distribution: Web Link Analysis**

- **PageRank** over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page (dotted lines, not all shown)
    - With prob. 1-c, follow a random outlink (solid lines)

- **Stationary distribution**
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors
Application of Stationary Distributions: Gibbs Sampling*

- Each joint instantiation over all hidden and query variables is a state: \( \{x_1, \ldots, x_n\} = H \cup Q \)

- Transitions:
  - With probability \( \frac{1}{n} \) resample variable \( x_j \) according to
    \[
P(X_j | x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, e_1, \ldots, e_m)
    \]

- Stationary distribution:
  - Conditional distribution \( P(X_1, X_2, \ldots, X_n | e_1, \ldots, e_m) \)
  - Means that when running Gibbs sampling long enough we get a sample from the desired distribution
  - Requires some proof to show this is true!
Hidden Markov Models
Hidden Markov Models

- Markov chains not so useful for most agent
  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states $X$
  - You observe outputs (effects) at each time step
Example: Weather HMM

An HMM is defined by:
- Initial distribution: $P(X_1)$
- Transitions: $P(X_t | X_{t-1})$
- Emissions: $P(E_t | X_t)$
Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process: future depends on past via the present
  - Current observation independent of all else given current state

- Quiz: does this mean that evidence variables are guaranteed to be independent?
  - [No, they tend to correlated by the hidden state]
Real HMM Examples

- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options

- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t | e_1, ..., e_t)$ (the belief state) over time.

- We start with $B_1(X)$ in an initial setting, usually uniform.

- As time passes, or we get observations, we update $B(X)$. 
Example: Robot Localization

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.
Example: Robot Localization

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

\[ t=1 \]
Example: Robot Localization

\[ t=2 \]
Example: Robot Localization
Example: Robot Localization

$\text{Prob} \quad 0 \quad 1$

$t=4$
Example: Robot Localization

$t=5$
The Forward Algorithm

- We are given evidence at each time and want to know

\[ B_t(X) = P(X_t|e_{1:t}) \]

- Induction: assuming we have current belief \( B(X_t) = P(X_t|e_{1:t}) \)

\[
P(X_{t+1}|e_{1:(t+1)}) \leftarrow P(X_{t+1}|e_{1:t}) \leftarrow P(X_t|e_{1:t})
\]

Observation update  Passage of time update
Inference: Base Cases

\[ P(X_1 | e_1) \]
\[ P(x_1 | e_1) = \frac{P(x_1, e_1)}{P(e_1)} \]
\[ \propto_{x_1} P(x_1, e_1) \]
\[ = P(x_1)P(e_1 | x_1) \]

\[ P(X_2) \]
\[ P(x_2) = \sum_{x_1} P(x_1, x_2) \]
\[ = \sum_{x_1} P(x_1)P(x_2 | x_1) \]
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$
  $$B(X_t) = P(X_t|e_{1:t})$$
- Then, after one time step passes:
  $$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$
  $$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$
  $$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$
- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step $t$ the belief is about, and what evidence it includes

Or compactly:
$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$
Observation

- Assume we have current belief $P(X \mid \text{previous evidence})$:
  
  $$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

- Then, after evidence comes in:
  
  $$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})}$$
  
  $\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$

  $$= P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

  $$= P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

- Or, compactly:
  
  $$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} \mid X_{t+1}) B'(X_{t+1})$$

- Basic idea: beliefs “rewighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize
Example: Weather HMM

\[ R_t R_{t+1} P(R_{t+1} | R_t) \]

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Online Belief Updates

- Every time step, we start with current $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

- We update for evidence:

$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$
Next Time: Particle Filtering and Applications of HMMs