Many slides are by Michal Jakob, Czech Technical University
Reminder:

- Written assignment 1
  - Deadline: May 01, 2020
Motivating Example: Ride Sharing

- People travelling between locations and would like to share a ride
  - Someone can pick up others on their way to their destination
  - Others have to go out of their way to pick up others
  - A car can hold only 5 people
- Factors: money and time
- Who should rideshare together?
- How much should they pay each other?
Concerns

- Rationality
  - Money versus time

- Fairness
  - Savings in money and loses in time should be fairly distributed

- Coalitional game theory formalizes such notions and provides techniques for working with them
Coalition (Team) Formation

- **Example**
  - Friends agreeing on a trip
  - Entrepreneurs trying to form companies
  - Companies cooperating to handle a large contract

- **Assumption**
  - A coalition can achieve more than (the sum of) individual agents
    - Better to team up and split the payoff than receive payoff individually

- **Coalitional (cooperative) game theory**
  - Agents till pursue their own interests
Non-Cooperative vs Coalitional (Cooperative)

Non-cooperative game theory
- Payoffs go directly to individual agents
- Players choose an action
- Model of strategic confrontation

Coalitional game theory
- Payoffs go to coalitions which redistribute them to their members
- Players choose a coalition to join and agree on payoff distribution
- Model of team / cooperation formation

Players are self-interested
Example: Task Allocation

- A set of tasks needs to be performed requiring different types of expertise and resources

- Individual agent do **not** have enough resources to perform all tasks
  - Need to team up

- Example
  - Transport domain: airline alliances
Example: Buying Ice Cream

- **N children:**
  - Each has some amount of money

- **Three types** of ice cream tubs:
  - Type 1 costs $7, contains 500g
  - Type 2 costs $9, contains 750g
  - Type 3 costs $11, contains 1kg

- Children only care about ice cream.

- **Payoff** of each group: the **maximum quantity** of ice cream the group can buy by pooling their money

- The ice cream can be **shared arbitrarily** within the group
How is a Coalitional Game Played?

- Payoffs for different coalitions are known
- Agents decide
  - Which coalitions
  - Which payoff distributions
  - Goal: benefits
- Agents agree on coalitions and payoff distributions
  - Require contracts – infrastructure for cooperation
- Task is executed and the payoff is distributed
Basic Definitions
Transferable Utility

- **Transferable utility**: the payoff to a coalition can be freely redistributed among its members
  - Satisfied whenever there is a universal currency that is used for exchange in the system
  - Means that each coalition can be assigned a single value as its payoff.

- Coalitional game with transferable utility
  - A set of players: \( N \)
  - Valuation function: each coalition \( C \subseteq N \) has a payoff \( v(C) \) which can be distributed among members.
Example

- **Valuation function**
  - $v(\emptyset)=v(\{\text{Charlie}\})=v(\{\text{Marcie}\})=v(\{\text{Pattie}\})=0$
  - $v(\{\text{Charlie, Marcie}\})=500$, $v(\{\text{Charlie, Pattie}\})=500$, $v(\{\text{Marcie, Pattie}\})=0$
  - $v(\{\text{Charlie, Marcie, Pattie}\})=750$
Outcome and Payoff Vector

- An outcome of a game \((N, \nu)\) is a pair of:
  - Coalitional structure (CS): a partition of \(N\) into coalitions \((C_1, C_2, \ldots, C_k)\)
  - Payoff (distribution) vector \((x_1, x_2, \ldots, x_N)\) for agents
    - Value of each coalition is distributed to its members: \(\sum_{i \in C} x_i = \nu(C)\)

- Payoff is *individually rational* if each agent is in favor of forming coalitions: \(x_i \geq \nu(\{\text{agent}_i\})\)
## Example

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<tr>
<th>Coalitions</th>
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- **Outcome:** \{1\}, \{2\}, \{3\}
  - \(x_1 = \) \(x_2 = \) \(x_3 = \)

- **Outcome:** \{1\}, \{2,3\}
  - \(x_1 = \) \(x_2 = \) \(x_3 = \)

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  - \(x_1 = \) \(x_2 = \) \(x_3 = \)

- **Outcome:** \{1,2,3\}
  - \(x_1 = \) \(x_2 = \) \(x_3 = \)
Super-Additive Games

- Sum of payoffs of any two disjoint coalitions is less than payoff of the union of these two coalitions
- Any disjoint coalitions $C$ and $D$ of agent set $N$:
  - $v(C \cup D) \geq v(C) + v(D)$

- Example: ice cream game
- Two agents can always merge without losing money
- Players would form the grand coalition (set of all agents)
Solution Concepts
Outcome Criteria

- **Fairness**
  - How well payoffs reflect each agent’s contribution?

- **Stability**
  - What are the incentives for agents to stay in a coalition structure?
Outcome Criteria

- **Fairness**
  - How well payoffs reflect each agent’s contribution?
  - *Shapley value* solution concept

- **Stability**
  - What are the incentives for agents to stay in a coalition structure?
  - *Core* solution concept
Outcome Criteria

- **Fairness**
  - How well payoffs reflect each agent’s contribution?
  - *Shapley value* solution concept

- **Stability**
  - What are the incentives for agents to stay in a coalition structure?
  - *Core solution concept*
Distributing Payments

- How should we fairly distribute a coalition’s payoff?

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- If the agents form \{1,2\}, how much should each get paid?
Fairness: Axiomatic Approach

1. **Symmetry**: if two players contribute the same, they should receive the same payoff

- Two players (i,j) are **interchangeable** if they always contribute the same amount to every coalition of other players. That is, 
  \[ v(C \cup \{i\}) = v(C \cup \{j\}) \]  
  for all coalitions \( C \) such that \( i, j \notin C \)

- Interchangeable agents receive the same payoff: \( x_i = x_j \)
Fairness: Axiomatic Approach

2. **Dummy player**: players that do not add value to any coalition should get what they earn on their own

- Player $i$ is a dummy player if the amount $i$ contributes to any coalition is exactly the amount $i$ is able to achieve alone. That is, $\nu(C \cup \{i\}) - \nu(C) = \nu(i)$ for all coalitions $C$ such that $i \notin C$

- Dummy player’s payoff: $x_i = \nu(\{i\})$
3. **Additivity**: if two games are combined, the value a player gets should be the *sum of the values* it gets in individual games.

- Consider two different coalitional games with valuation functions $v', v''$, involving the same set of players.
- If we combine them into a single game such that each coalition $C$ receives the payoff $v^+(C) = v'(C) + v''(C)$, then players’ payoff in the new game is $x_i^+ = x_i' + x_i''$, $\forall i$. 
Theorem: Given a coalitional game \((N, v)\), there is a unique payoff division \(\phi(N, v)\) that divides the full payoff of the grand coalition and satisfies the Symmetry, Dummy player, and Additivity axioms.

- This payoff division is called Shapley Value

Lloyd S. Shapley. 1923-2016. Responsible for the core and Shapley value solution concepts.
**Shapley Value**

**Definition:** Given a coalitional game \((N, v)\), the Shapley value of player \(i\) is given by:

\[
\phi_i(N, v) = \frac{1}{|N|!} \sum_{C \subseteq N \setminus \{i\}} |C|! \times (|N| - |C| - 1)! \times [v(C \cup \{i\}) - v(C)]
\]

- **Intuition:** Shapley value captures the average marginal contribution of player \(i\)
  - Averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition
Example

- Grand coalition \{1,2\}

- Shapley value:
  - \(\phi_1 = \frac{1}{2}(v(\{1\}) - v(\emptyset) + v(\{2,1\}) - v(\{2\})) = 2\)
  - \(\phi_2 = \frac{1}{2}(v(\{2\}) - v(\emptyset) + v(\{2,1\}) - v(\{1\})) = 4\)
Example

Valuation function

- $v(\emptyset) = v(\{\text{Charlie}\}) = v(\{\text{Marcie}\}) = v(\{\text{Pattie}\}) = 0$
- $v(\{\text{Charlie, Marcie}\}) = 500$, $v(\{\text{Charlie, Pattie}\}) = 500$, $v(\{\text{Marcie, Pattie}\}) = 0$
- $v(\{\text{Charlie, Marcie, Pattie}\}) = 750$

Shapley value for Charlie

- $\phi_C = \frac{1}{3!} \left( 2 \times (v(C) - v(\emptyset)) + v(C, M) - v(M) + v(C, P) - v(P) + 2 \times (v(C, M, P) - v(M, P)) \right) = 416 \frac{2}{3}$
Outcome Criteria

- **Fairness**
  - How well payoffs reflect each agent’s contribution?
  - Shapley value solution concept

- **Stability**
  - What are the incentives for agents to stay in a coalition structure?
  - Core solution concept
What is a Good Outcome?

- Valuation function
  - $v(\emptyset)=v(\{\text{Charlie}\})=v(\{\text{Marcie}\})=v(\{\text{Pattie}\})=0$
  - $v(\{\text{Charlie, Marcie}\})=500$, $v(\{\text{Charlie, Pattie}\})=500$, $v(\{\text{Marcie, Pattie}\})=0$
  - $v(\{\text{Charlie, Marcie, Pattie}\})=750$

- How should players share ice cream?
  - What about the sharing (Charlie: 200, Marcie: 200, Pattie: 350)?
  - Not stable (Charlie and Marcie can get more ice cream by buying a 500g tub on their own and splitting it equally)
Core

- Under what payoff distributions is the outcome of the game stable?
  - Each sub-coalition earns at least as much as it can make on it owns

- A payoff vector $\vec{x}$ is in the core of a coalitional game $(N, v)$ if and only if:
  - For all coalitions $C$: $\sum_{i \in C} x_i \geq v(C)$

- Core: the set of all stable outcomes
  - No coalition wants to deviate from
Ice Cream Game: Core

Valuation function

- $v(\emptyset) = v(\{\text{Charlie}\}) = v(\{\text{Marcie}\}) = v(\{\text{Pattie}\}) = 0$
- $v(\{\text{Charlie, Marcie}\}) = 500$, $v(\{\text{Charlie, Pattie}\}) = 500$, $v(\{\text{Marcie, Pattie}\}) = 0$
- $v(\{\text{Charlie, Marcie, Pattie}\}) = 750$

- Is (200, 200, 350) in the core?
- Is (250, 250, 250) in the core?
- Is (750, 0, 0) in the core?
  - Not very fair!!!
Existence of Stable Outcome

- Not always exist

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Other Core Concepts

- \( \varepsilon \)-core of a super-additive coalitional game
  - Each sub-coalition earn at least as much as it can make on it owns minus \( \varepsilon \)
  - For all coalitions \( C \): \( \sum_{i \in C} x_i \geq v(C) - \varepsilon \)

- Least core
  - Is \( \varepsilon^* \)-core where \( \varepsilon^* \) is the smallest number such that the \( \varepsilon^* \)-core is non-empty
Summary

- Coalitional game theory models the formation of teams of selfish agents

- Shapley value solution concept: represents a fair distribution of payments

- Core solution concept: address the issue of coalition stability