CIS 410/510 (Spring 2019): Multi-Agent Systems and Real-world Applications
Lecture 7: Extensive-Form Games

Thanh H. Nguyen
University of Oregon
Recall: Extensive-Form Games with Perfect Information

A perfect-information game is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$

- $N$ is a set of N players
- $A$ is a set of actions
- $H$ is a set of non-terminal choice nodes
- $Z$ is a set of terminal nodes, disjointed from $H$
- $\chi: H \mapsto 2^A$ the action function, which assigns to each choice node a set of possible actions
- $\rho: H \mapsto N$ is the player function, which assigns to each choice node a player $i \in N$ who choose an action at that node
- $\sigma: H \times A \mapsto H \cup Z$ is the successor function, which maps a choice node and an action to a new choice node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
- $u = (u_1, u_2, ..., u_n)$ where $u_i: Z \mapsto \mathbb{R}$ is a real-valued utility function for player $i$ on the terminal nodes $Z$
Recall: Extensive-Form Games with Perfect Information

<table>
<thead>
<tr>
<th>Strategies</th>
<th>(C,E)</th>
<th>(C,F)</th>
<th>(D,E)</th>
<th>(D,F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A,G)</td>
<td>3, 8</td>
<td>3, 8</td>
<td>8, 3</td>
<td>8, 3</td>
</tr>
<tr>
<td>(A,H)</td>
<td>3, 8</td>
<td>3, 8</td>
<td>8, 3</td>
<td>8, 3</td>
</tr>
<tr>
<td>(B,G)</td>
<td>5, 5</td>
<td>2, 10</td>
<td>5, 5</td>
<td>2, 10</td>
</tr>
<tr>
<td>(B,H)</td>
<td>5, 5</td>
<td>1, 0</td>
<td>5, 5</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
Recall: Extensive-Form Games with Perfect Information

<table>
<thead>
<tr>
<th>Strategies</th>
<th>(C,E)</th>
<th>(C,F)</th>
<th>(D,E)</th>
<th>(D,F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A,G)</td>
<td>3, 8</td>
<td>3, 8</td>
<td>8, 3</td>
<td>8, 3</td>
</tr>
<tr>
<td>(A,H)</td>
<td>3, 8</td>
<td>3, 8</td>
<td>8, 3</td>
<td>8, 3</td>
</tr>
<tr>
<td>(B,G)</td>
<td>5, 5</td>
<td>2, 10</td>
<td>5, 5</td>
<td>2, 10</td>
</tr>
<tr>
<td>(B,H)</td>
<td>5, 5</td>
<td>1, 0</td>
<td>5, 5</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
Imperfect Information Extensive-Form Games

- **Perfect information**
  - All players know the game structure
  - Each player, when making decision, is perfectly informed of all events that have previously occurred

- **Imperfect information**
  - All players know the game structure
  - Partial or no knowledge of actions taken by others
  - Limited memory of own past actions (imperfect recall)
Example

- Player 1 does not know actions taken by player 2
- Nodes $n_4$ and $n_5$ are not distinguishable to player 1
Imperfect Information Game

- Extensive form game

- Each player’s choice nodes are partitioned into information sets

- Intuitively, if two choice nodes are in the same information set then the agent cannot distinguish between them.

- Decisions are made based on information sets
Imperfect-Information Game (Extensive Form)

- **Information sets** of player $i$: $I_i = (I_{i,1}, I_{i,2}, \ldots, I_{i,k_i})$ is a partition of chance nodes of player $i$ such that: any chance nodes belonging to the same information set, i.e., $h \in I_{i,j}$ and $h' \in I_{i,j}$
  - Action function $\chi(h) = \chi(h')$: same set of actions
  - Player function $\rho(h) = \rho(h') = i$: same player
Example

- Information sets of player 1

- Information sets of player 2
Strategies in Extensive Form Games (EFGs) with Imperfect Information

- Complete assignments from information sets to actions

- Player 1
  - {\( (L, l) \), \( (L, r) \), \( (R, l) \), \( (R, r) \) }

- Player 2
  - {A, B}
Strategies in Extensive Form Games (EFGs) with Imperfect Information

- A pure strategy of a player: a complete specification of which deterministic action to take at every information set belonging to that player.
Strategies in Extensive Form Games (EFGs) with Imperfect Information

- A mixed strategy
  - A probability distribution over pure strategies
  - Player 1: (L, l): 0.1, (L, r): 0.3, (R, l): 0.4, (R, r): 0.2

- A behavioral strategy
  - A product of independent probability distributions over actions in each information set
  - Player 1: (L: 0.4, R: 0.6), (l: 0.25, r: 0.75)
There is no guarantee that a pure-strategy Nash equilibrium exists in imperfect information games.
Perfect Recall vs Imperfect Recall

- **Perfect Recall**
  - No player forgets anything he knew about moves made so far

- **Imperfect Recall**
  - Not exhibit perfect recall
Perfect Recall

Player 1

\[ n_1 \]

\[ n_2 \]

\[ n_3 \]

Player 2

\[ n_4 \]

\[ n_5 \]

\[ n_6 \]

\[ n_7 \]

\[ n_8 \]

\[ n_9 \]

\[ n_{10} \]

\[ n_{11} \]

\[ n_{12} \]

\[ n_{13} \]

\[ n_{14} \]

\[ n_{15} \]

Actions:

- Player 1: A, B
- Player 2: A, B

Payoffs:

- (1, -1)
- (-1, 1)
- (-5, 5)
- (1, -1)
Imperfect Recall

Player 1

L

R

Player 1

Player 2

(1, -1)

(-1, 1)

(1, -1)

(-5, 5)

(-5, 5)

(-1, 1)
Properties of Perfect Recall

Player 1

Player 2

Player 1

Player 2

Player 1

Player 1
Properties of Perfect Recall

- Paths to nodes in the same information set of a player
  - Same length
  - Same sequence of information sets
  - Same sequence of actions taken by the player

- A behavioral strategy is equivalent to a mixed strategy
  - Induce same probabilities on outcomes
Sequence Form

- **Sequence (of actions)** of a player corresponding to a node $h$:
  - An ordered set of player’s actions on the path from root to $h$

- **Example:**
  - Sequences of player 1
    - $\emptyset, L, R, Ll, Lr$
  - Sequences of player 2
    - $\emptyset, A, B$
**Sequence Form: Payoff Function**

- **Payoffs of players:**
  - **Utilities at leaf node** which can be reached when each player plays his sequence
  - 0 if no leaf node can be reached.

<table>
<thead>
<tr>
<th></th>
<th>$\emptyset$</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>L</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>R</td>
<td>1, 1</td>
<td>1, 1</td>
<td>1, 1</td>
</tr>
<tr>
<td>Ll</td>
<td>0, 0</td>
<td>0, 0</td>
<td>2, 4</td>
</tr>
<tr>
<td>Lr</td>
<td>0, 0</td>
<td>2, 4</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Realization Plan

- Mapping sequences to probabilities that recovers players’ behavioral strategies

- Realization plan of player 1

  \[
  \begin{align*}
  r_1(\emptyset) &= 1, \quad r_1(L) + r_1(R) = r_1(\emptyset) \\
  r_1(Ll) + r_1(Lr) &= r_1(L) \\
  r_1(\emptyset), \ r_1(L), \ r_1(R) &\geq 0 \\
  r_1(Ll), \ r_1(Lr) &\geq 0
  \end{align*}
  \]

- Behavioral strategy of player 1

  \[
  \begin{align*}
  \beta_1(L) &= r_1(L), \quad \beta_1(R) = r_1(R) \\
  \beta_1(l) &= \frac{r_1(Ll)}{r_1(L)}, \quad \beta_1(r) = \frac{r_1(Lr)}{r_1(L)}
  \end{align*}
  \]