Transitive Closure

Let adjacency matrix be defined as usual

\[ A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & \text{otherwise} \end{cases} \]

The transitive closure of \( A \) is \( A^* \) where

\[ A^*_{ij} = \begin{cases} 1 & \text{if there is a path from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases} \]

Recall

\[ A^k \text{ contains info about paths of length } k \]

\[ A^k[i,j] = \begin{cases} \text{number of paths from } i \text{ to } j \text{ of length } k \end{cases} \]

Modify + becomes OR

\[ A^2[i,j] = \begin{cases} 1 & \text{if there is a path of length 2 from } i \text{ to } j \\ 0 & \text{no path of length 2} \end{cases} \]

Now \( A^k[i,j] = \begin{cases} 1 & \text{if there is a path of length } k \text{ from } i \text{ to } j \\ 0 & \text{no path of length } k \end{cases} \)

Since a path from \( i \) to \( j \) would have length \( k \) for

\[ k = 0, 1, 2, \ldots, n-1 \]

\[ A^* = I + A + A^2 + A^3 + \cdots + A^{n-1} \]
Note In our new world of $+, \times$ multiplication by constants does nothing

\[
\begin{align*}
5 \cdot 1 &= 1 + 1 + 1 + 1 + 1 = 1 \\
5 \cdot 0 &= 0
\end{align*}
\]

$$5 \cdot A = A$$

We only care about

$\mathbf{0} = "\text{num}"$

$\mathbf{1} = "\text{not 0}"$

Observe

\[(I + A)^{n-1} \]

\[
= \begin{pmatrix}
\binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-1}
\end{pmatrix}
\]

\[
= A^0 + A^1 + A^2 + \cdots + A^{n-1}
\]

\[= A^*\]

Slow way to compute

\[
A^* = \mathbf{0}
\]

\[
L = \begin{pmatrix} i \end{pmatrix} \text{I}
\]

for $i = 1$ to $n$

\[
A^3 + = L \quad \leftarrow O(n^3)
\]

\[
L \times = A \quad \leftarrow O(n^3)
\]

\[
O(n^{3} \lg n)
\]

 Faster \ (I+A)^{n-1} \hspace{1cm} \text{using repeated squaring}

\[O(n^3 \lg n)\]

\[\sim O(n^{3.81} \lg n) \hspace{1cm} \text{via Strassen} \]
To get the length of the shortest paths

Given matrix $W$

$w_{ij}$ - distance from $i$ to $j$

Compute $L^m$

$L^m_{ij}$ = length of shortest path from $i$ to $j$

$L^1 = W$

To get $L' = (L^m)$ from $L = (L^\infty)$, $L' = L'W$

Extend Shortest Path

for $i = 1$ to $n$

for $j = 1$ to $n$

$l'_{ij} = \infty$

for $k = 1$ to $n$

$l'_{ij} = \min(l'_{ij}, l'_{ik} + w_{kj})$

Then $L = L'$
or return $L'$

Try all vertices $k$

"length $m-1$ path to $k$" directly from $k$ to $j"
Compute to

\underline{Matrix Multiply}

\begin{align*}
\text{for } i = 1 \text{ to } n \\
\text{for } j = 1 \text{ to } n \\
\text{for } k = 1 \text{ to } n \\
c_{ij} &= 0 \\
c_{ij} &= c_{ij} + a_{ik} \cdot b_{kj}
\end{align*}

\begin{align*}
\text{ESP} \\
\ell^{m-1} & \quad a \\
w & \quad b \\
\ell^m & \quad c \\
\text{MIN} & \quad + \\
+ & \quad \cdot
\end{align*}

Thus

\[ O(n^3 \log n) \text{ APSP method} \]

(to get \( L^{n-1} \))
Note: The use of a subproblem

all paths of length 1
all paths of length 2
all paths of length 3
\[ \vdots \]
all paths of length \( n-1 \)

Useful to look at other subproblems:

restrict intermediate vertices. Let \( S \subseteq V \) be a subset of vertices.

look at all paths from \( i \) to \( j \) with intermediate vertices in \( S \)

\[ i \rightarrow u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_k \rightarrow j \]

\( u_i \in S \)

find the shortest such path

length of path doesn't matter (though it's at most \( |S| + 1 \))

Floyd-Warshall

find shortest path of all paths with intermediate vertices in \( S \)

\[ \{1 \} \]
\[ \{1, 2 \} \]
\[ \{1, 2, 3 \} \]
\[ \{1, 2, 3, \ldots, n \} \]