TAKE-HOME FINAL EXAM

due Thursday, December 12, 2019

INSTRUCTIONS Undergraduates should do 5 of the following questions, and graduate students should do 7.

1. Write a PDA $P$ for the language $\{a^j b^k c^j | j, k \geq 0\}$. Make it as small as possible, subject to the following constraints:
   - It has a single accept state
   - it empties the stack before accepting
   - each transition is a push or a pop

Then, use the construction of lemma 2.27 to construct a CFG directly from $P$.

2. Pumping lemma:
   (a) Show that $A = \{ w \in \{a, b, c\}^* | w$ has more a’s than b’s $\}$ is not regular.
   (b) Show that $B = \{ w | w \in \{a, b, c\}^*, n_a(w)/n_b(w) = n_c(w) \}$ is not context-free. (Here, $n_a(w)$ means the number of a’s in $w$, similarly for $n_b(w)$ and $n_c(w)$.)
   (c) Show that $C = \{ a^i | i \geq 0 \}$ is not context free

3. Convert the NFA of figure 1 to a DFA. The start state is $q_0$, the accepting set is $F = \{q_3\}$, and “epsilon” means $\epsilon$.

4. Let $A$ and $B$ be two disjoint languages. Say that language $C$ separates $A$ and $B$ if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two co-Turing-recognizable languages are separable by some decidable language.

5. Let $S = \{ \langle M \rangle | M$ is a DFA that accepts $w^R$ whenever it accepts $w \}$. Show that $S$ is decidable.

6. Build some context free items:
   (a) Construct a PDA $M$ (show diagram) such that
   $$L(M) = \{ ax^nby^mcz^{2m}dx^n | m, n \geq 0 \}.$$
   (b) Show a CFG $G$ such that
   $$L(G) = \{ x^n # y^m | 0 \leq 2m \leq n \leq 4m \}.$$

7. Consider the grammar $G$ given by $S \rightarrow aSb|bSa|SS|\epsilon$. We want to show very carefully that $L(G) = A$ where $A = \{ w \in \{a, b\}^* | w$ contains an equal number of a’s and b’s $\}$. 
(a) Prove the following: (Claim 1) if \( w \in A \) and \( w = axb \) or \( w = bxa \), then \( x \in A \).

(b) Prove the following: (Claim 2) if \( w \in A \) and \( w = axa \) or \( w = bxb \), then there are strings \( y, z \in A \) such that \( w = yz \).

(c) Prove by induction on the length of \( w \) that if \( w \in A \) then there is a derivation \( S \Rightarrow w \).

(d) Argue that if there is a derivation \( S \Rightarrow w \), then \( w \) has an equal number of a’s and b’s.

**Reductions and Completeness**

We have defined a \( m \)-reduction (here \( m \) means “mapping” or “many-one”) from language \( A \) and to \( B \) as \( A \leq_m B \) iff there is a computable string function \( f : \Sigma^* \rightarrow \Sigma^* \) satisfying

\[
\forall w \in \Sigma^*, w \in A \iff f(w) \in B.
\]

Furthermore, we define a language \( K \) to be \( m \)-complete (formally “complete for the Turing-recognizable sets under \( \leq_m \)”) if (i) \( K \) is Turing-recognizable and (ii) for any Turing-recognizable language \( A, A \leq_m K \). Note that in class we saw that both \( A_{TM} \) and \( HALT_{TM} \) are \( m \)-complete.

8. Show the following

(a) If \( A \leq_m B \) and \( B \leq_m C \), then \( A \leq_m C \).

(b) If \( A \leq_m B \) and \( B \) is Turing-decidable, then \( A \) is Turing-decidable.

(c) If \( A \leq_m B \) and \( B \) is Turing-recognizable, then \( A \) is Turing-recognizable.

(d) If \( A \leq_m B \), \( B \) is Turing-recognizable, and \( A \) is \( m \)-complete, then \( B \) is \( m \)-complete.

**A Computational Hierarchy via Alternating Quantifiers**

We say that a set \( A \) is \( \Sigma_k \) if it can be characterized as

\[
A = \{ x | \exists y_1 \forall y_2 \exists y_3 \ldots Q_k y_k \langle x, y_1, y_2, y_3, \ldots, y_k \rangle \in B \}
\]

where \( B \) is decidable and quantifier \( Q_k = \exists \) if \( k \) is odd and \( \forall \) if it is even. (Here \( x \) and all the \( y_i \)'s are strings over the same alphabet.) Similarly, \( A \) is \( \Pi_k \) if we can write

\[
A = \{ x | \forall y_1 \exists y_2 \forall y_3 \ldots Q_k y_k \langle x, y_1, y_2, y_3, \ldots, y_k \rangle \in B \}
\]

where \( B \) is decidable and quantifier \( Q_k = \forall \) if \( k \) is odd and \( \exists \) if it is even.

**Other definitions:**

- We can also say that \( A \) is \( \Sigma_k \) if it is Turing-recognizable with an oracle from \( \Sigma_{k-1} \)
- We define \( A \) as \( \Delta_k \) if it is Turing-decidable with an oracle from \( \Sigma_{k-1} \). (Equivalently, \( A \leq_T B \) for some \( B \in \Sigma_{k-1} \).)

**Observations:**

- the decidable languages are \( \Sigma_0 = \Pi_0 = \Delta_1 \) (these are the base cases)
• the recognizable languages are $\Sigma_1$
• if a set $A$ is $\Sigma_k$ then its complement $\overline{A}$ is $\Pi_k$
• for example $TOT_{TM} = \{ \langle M \rangle \mid L(M) = \Sigma^* \}$ is $\Pi_2$ since we can write $TOT_{TM} = \{ \langle M \rangle \mid \forall w \exists t \langle M, w, t \rangle \in B \}$ where $B = \{ \langle M, w, t \rangle \mid M$ accepts $w$ in at most $t$ steps $\}$

9. (a) Show that if $A$ is $\Sigma_k$ then it is both $\Sigma_{k+1}$ and $\Pi_{k+1}$.
(b) Show that if $A$ is both $\Sigma_k$ and $\Pi_k$, then $A$ is $\Delta_k$

10. Give a $\Sigma_k$ or $\Pi_k$ characterization of the following problems
(a) $INF_{TM} = \{ \langle M \rangle \mid L(M)$ has an infinite number of strings $\}$
(b) $COF_{TM} = \{ \langle M \rangle \mid$ the complement of $L(M)$ has a finite number of strings $\}$
(c) $ETM = \{ \langle M \rangle \mid L(M) = \emptyset \}$
(d) $CFL_{TM} = \{ \langle M \rangle \mid L(M)$ is a CFL $\}$

Figure 1: NFA for problem 3