Assignment 3

due Monday, Feb 5, 2018

1. Consider the graph below. You will be building a MST for this graph in two ways. When there is a tie on the edge weights, consider the edges or nodes in alphabetical order. (For an edge, this means (a,f) before (b,c) and (c,d) before (c,g).)
   
   (a) Use Kruskal’s method.
   (b) Use Prim’s method.

![Figure 1: for question 1]

[8 points]

2. Suppose you are given the diagram of a telephone network, which is a graph \( G \) whose vertices represent switching centers and whose undirected edges represent communication lines between two centers. The edges \( (u, v) \) are marked by their bandwidth \( B[u, v] \). The bandwidth of a path is the bandwidth of its lowest bandwidth edge. Give an algorithm that, given a diagram and two switching centers \( s \) and \( t \), will output the bandwidth of a path with maximum bandwidth between \( s \) and \( t \). [6 points]

3. We are given a graph \( G = (V, E) \) where \( V \) represents a set of locations and \( E \) represents a communications channel between two points. We are also given locations \( s, t \in V \), and a
reliability function $r : V \times V \to [0, 1]$. You need to give an efficient algorithm which will output the reliability of the most reliable path from $s$ to $t$ in $G$.

For any points $u, v \in V$, $r(u, v)$ is the probability that the communication link $(u, v)$ will not fail: $0 \leq r(u, v) \leq 1$. Note that if there is a path with two edges, for example, from $u$ to $v$ to $w$, then the reliability of that path is $r(u, v) \cdot r(v, w)$. [6 points]

4. Suppose we are given a directed graph $G$ with $n$ vertices, and let $M$ be the $n \times n$ adjacency matrix corresponding to $G$.

(a) Let the product of $M$ with itself ($M^2$) be defined, for $1 \leq i, j \leq n$, as follows:

$$M^2(i, j) = M(i, 1) \odot M(1, j) \oplus \cdots \oplus M(i, n) \odot M(n, j),$$

where $\oplus$ is the logical or operator and $\odot$ is the logical and operator. Given this definition, what does $M^2(i, j) = 1$ imply about the vertices $i$ and $j$? What if $M^2(i, j) = 0$?

(c) Now suppose that $G$ is weighted and assume the following

- for $1 \leq i \leq n$, $M(i, i) = 0$
- for $1 \leq i, j \leq n$, $M(i, j) = \text{weight}(i, j)$ if $(i, j) \in E$
- for $1 \leq i, j \leq n$, $M(i, j) = \infty$ if $(i, j) \notin E$

Also let $M^2(i, j)$ be defined, for $1 \leq i, j \leq n$, as follows:

$$M^2(i, j) = \min[M(i, 1) + M(1, j), \ldots, M(i, n) + M(n, j)].$$

If $M^2(i, j) = k$, what may we conclude about the relationship between vertices $i$ and $j$.

[6 points]

5. exercise 24.1-3 from CLRS

Given a weighted, directed graph $G = (V, E)$ with no negative-weight cycles, let $m$ be the maximum over all vertices $v \in V$ of the minimum number of edges in a shortest path from the source $s$ to $v$. (Here, the shortest path is by weight, not the number of edges.) Suggest a simple change to the Bellman-Ford algorithm that allows it to terminate in $m + 1$ passes, even if $m$ is not known in advance. [4 points]

Total: 30 points