Assignment 2

You can either submit it electronically to Canvas or bring a physical copy to the lecture. This assignment is 12% in your total points. For the simplicity of the grading, the total points for the assignment is 60.

1 Problems that you don’t need to submit solutions

However, you probably want to figure out answers to these problems as well.

Problem 1.1.

• Write down a matrix that is not Hermitian.

• Let $A \in L(\mathbb{C}^d)$ be a Hermitian matrix. Prove that if for all $|\psi\rangle \in \mathbb{C}^d$, $\langle \psi | A | \psi \rangle \geq 0$, then $A$ has only non-negative eigenvalues.

• Let $A \in L(\mathbb{C}^d)$ be a Hermitian matrix. Prove that if $A$ has only non-negative eigenvalues, then for all $|\psi\rangle \in \mathbb{C}^d$, $\langle \psi | A | \psi \rangle \geq 0$.

Problem 1.2. Given $|-\rangle$ state, and suppose that we measure in the computational basis $B = \{|0\rangle \langle 0|, |1\rangle \langle 1|\}$. What are the probabilities for each possible measurement outcome, and the corresponding post-measurement states?

Problem 1.3.

• Let $A, B \in L(\mathbb{C}^d)$ be positive semi-definite matrices. Prove that $A + B$ is positive semi-definite.

• Prove that if $\rho$ and $\sigma$ are density matrices, then so is $p_1 \rho + p_2 \sigma$ for any $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.

Problem 1.4. Let $|\psi\rangle = \alpha_0 |a_0\rangle |b_0\rangle + \alpha_1 |a_1\rangle |b_1\rangle$ be the Schmidt decomposition of a two-qubit state $|\psi\rangle$. Prove that for any single qubit unitaries $U$ and $V$, $|\psi\rangle$ is entangled if and only if $|\psi'\rangle = (U \otimes V) |\psi\rangle$ is entangled.
2 Problems that you’re required to submit solutions

Please only submit the solutions to problems in this section.

Problem 2.1 [10 points; 5 points each]. Entangled states. Determine whether the following state is a product state or an entangled state.

- \[ |\psi\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \]
- \[ |\psi\rangle = \frac{1}{2} (|00\rangle + i|01\rangle + i^2|10\rangle + i^3|11\rangle), i = \sqrt{-1} \]

Problem 2.2 [10 points; 5 points each]. Density Operators.

- Suppose with probability 1/3, you are given state |0\rangle and with probability 2/3, you are given state |+\rangle. Write down the density matrix (i.e., as a 2 \times 2 matrix) that describe the state in your possession.
- Let bipartite state |\psi\rangle = \alpha |01\rangle - \beta |00\rangle. Let \[ \rho = \frac{1}{2} |\Phi^+\rangle\langle\Phi^+| + \frac{1}{2} |\psi\rangle\langle\psi| \] Compute tr\(_B(\rho)\). Note that |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).

Problem 2.3 [10 points: 5 points each]. Measurements. Let

\[ |\psi\rangle = \frac{1}{2} (|00\rangle (\alpha |0\rangle + \beta |1\rangle) + |01\rangle (\alpha |1\rangle + \beta |0\rangle) + |10\rangle (\alpha |0\rangle - \beta |1\rangle) + |11\rangle (\alpha |1\rangle - \beta |0\rangle) \] .

Model the computational basis measurement on qubits 1 and 2 by \[ B = \{|ij\rangle\langle ij| \otimes I : i, j \in \{0, 1\}\} \], where the identity matrix \( I \) is on qubit 3.
- Show the probability of obtaining outcome \( i, j \) for any \( i, j \in \{0, 1\} \) is 1/4. (Hint: Try not to do the proof on a case-by-case basis for each possible \( i, j \).)
- Suppose that we instead perform the following projective measurement
  \[ B = \{(|00\rangle\langle00| + |11\rangle\langle11|) \otimes I, (|10\rangle\langle10| + |01\rangle\langle01|) \otimes I\} \].
  What is the post-measurement state \( \psi' \in (\mathbb{C}^2)^{\otimes 3} \) if we obtain the second outcome?

Problem 2.4 [15 points: 5 points each]. Quantum Circuits.

- Describe a two-qubit quantum circuit consisting of one CNOT gate and two Hadamard gates that computes the following unitary transformation:
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1 
  \end{pmatrix}
  \]
• Define the one-qubit gates $H$ (Hadamard) and $S$ as 

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad i = \sqrt{-1}. $$

In each case, give the $4 \times 4$ unitary matrix corresponding to the two-qubit controlled gate: (note that the dot denotes the control bit. when it is 0, do nothing on the target bit; otherwise, perform the corresponding gate.)

\[
\begin{array}{cc}
H & \\
\hline
H & S \\
\hline
S & S
\end{array}
\]

• Give the $4 \times 4$ unitary matrix corresponding to the following quantum circuit

\[
\begin{array}{cc}
H & \\
\hline
S & H
\end{array}
\]

where $S$ is as defined in part (b), and the last (two-qubit) gate denotes a swap gate, that transposes the two qubits (more precisely, a swap gate maps $|00\rangle \rightarrow |00\rangle$, $|10\rangle \rightarrow |01\rangle$, $|01\rangle \rightarrow |10\rangle$, and $|11\rangle \rightarrow |11\rangle$).

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**Problem 2.5 [15 points: 5 points each]. A qubit cannot be used to communicate a trit perfectly.**

Suppose that Alice wants to convey a trit of information (an element of $\{0, 1, 2\}$) to Bob and all she is allowed to do is prepare one qubit and send it to Bob. Bob is allowed to prepare $n-1$ additional qubits, each in state $|0\rangle$, and apply an $n$-qubit unitary $U$ operation to the entire $n$ qubit system followed by a measurement in the computational basis.

qubit from Alice

\[
H
\]

$U$

\[
S
\]

$H$

The outcome will be an element of $\{0, 1\}^n$. It is conceivable that such a scheme could exist where Bob can determine the trit from these $n$ bits (e.g., by a function $f(x_1, \cdots, x_n) \in \{0, 1, 2\}$). We shall prove that this is impossible.

The framework is that Alice starts with a trit $j \in \{0, 1, 2\}$ (unknown to Bob) and, based on $j$, prepares a one-qubit state, $\alpha_j |0\rangle + \beta_j |1\rangle, \, j \in \{0, 1, 2\}$, and sends it to Bob.

Then Bob applies some $n$-qubit unitary $U$ to $(\alpha_j |0\rangle + \beta_j |1\rangle) |00\cdots0\rangle$ and measures each qubit in the computational basis, obtaining some $x \in \{0, 1\}^n$ as outcome. Finally, Bob applies some function $f : \{0, 1\}^n \rightarrow \{0, 1, 2\}$ to $x$ to obtain a trit. The scheme works if and only if, starting with any $j \in \{0, 1, 2\}$, the resulting $x$ will satisfy $f(x) = j$ with probability 1.

• Note that each row of the matrix $U$ is a $2^n$-dimensional vector. For $j \in \{0, 1, 2\}$, define the space $V_j$ to be the span of all rows of $U$ that are indexed by an element of the set $f^{-1}(j) \subseteq \{0, 1\}^n$. Prove that $V_0, V_1, V_2$ are mutually orthogonal spaces.

• Explain why, for a scheme to work, $(\alpha_j |0\rangle + \beta_j |1\rangle)|00\cdots0\rangle \in V_j$ must hold for all $j \in \{0, 1, 2\}$.

• Prove that it is impossible for $(\alpha_j |0\rangle + \beta_j |1\rangle)|00\cdots0\rangle \in V_j$ to hold for all $j \in \{0, 1, 2\}$.