1. Add the following definitions to trees.ml. Do not use mutation. Do not change the type definitions or functions provided to you. Do not assume trees are sorted (though this is largely irrelevant).

(a) Define fromList of type int list -> inttree to make a sorted tree containing exactly the ints in the list without repeats. Use insert (provided).

(b) Define three functions, sum1, prod1, and avg1, to compute the sum, product, and average of the ints in a tree. Each has type inttree -> int. The sum and product of an empty tree are 0 and 1, respectively. For average, the empty tree should cause a DivisionByZero exception to be raised. For sum1 and prod1, do not use helper functions. For avg1, do not traverse the tree more than once. (Hint: Have a helper function do the traversal and return a pair.)

(c) Define map of type (int -> int) -> inttree -> inttree to produce a tree with the same shape as its second argument with the int at each position the result of applying the first argument to the int at the same position in the second argument.

(d) Define negateAll of type inttree -> inttree using map. It produces a tree of the same shape where each int is replaced with its negation.

(e) Define sum2, prod2, and avg2 to compute the sum, product, and average of a tree (see above), but using fold (provided). You should not need more than 1 line (possibly 2 for avg2). Use the same pair technique for average.

(f) In a short English paragraph, explain how a client of the iter function (provided) would use it to process all the ints in a tree. In a second short English paragraph, explain how iter is implemented (e.g., “when” and “how” it traverses the tree).

(g) Define sum3, prod3, and avg3 to compute the sum, product, and average of a tree (see above), but using iter (provided). For product, the code must “stop as soon as it sees a 0” (this is easier than when using fold). Hint: You should need about 5 lines for each function. For each, use a local helper function as a “loop” that takes the iterator and the answer-so-far.

(h) Challenge Problem: When writing functions that cannot always produce answers (e.g., average on an empty tree), one chooses between raising an exception or returning an option type. In this problem, you write short higher-order wrapper functions to “make the other choice”:

- Write optionToException of type ('a -> 'b option) -> exn -> 'a -> 'b. It should return a function that takes an argument and raises the exn if and only if the 'a -> 'b option returns None for the same argument.

- Write exceptionToOption of type ('a -> 'b) -> (exn -> bool) -> 'a -> 'b option. It should return a function that takes an argument and returns None if and only if the ('a -> 'b) (when passed the same argument) raises an exception that causes the exn -> bool to return true.

- In English, explain why optionToException takes an exn but exceptionToOption takes an exn -> bool.

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1 You might also try implementing iter using mutation instead of higher-order functions. It is not very pleasant.
2. We will extend IMP with the ability to push and pop heaps (material through lecture 3):

- A program state is now $L; H; s$ where $H$ and $s$ are as before and $L$ is a list (i.e., a stack of heaps).
- There are two new statement forms: `pushheap` and `popheap x`. The former adds the current heap $H$ to the beginning of $L$. (Informally, it “copies” $H$ “all at once”.) The latter replaces the current heap $H$ with the first element of $L$ except $x$ maps to $H(x)$ and replaces $L$ with the tail of $L$. If $L$ is the empty list, then `popheap x` has no effect. Both `pushheap` and `popheap x` “become skip” in one step.

(a) Extend the provided interpreter to support this new language. Hints/requirements:
- Do not use mutation.
- Change only `interp_step`, `iter`, and `interp`. (The parser is already written. The code for heaps and expressions does not need changing.)
- Have `interp_step` and `iter` take a third argument (the list of heaps) and have `interp_step` also return a “new” list.
- You need to change each case of `interp_step` and add two more cases corresponding to the two new statement forms (`pushheap` and `popheap x`).

(b) Give an IMP program using `pushheap` and `popheap x` that we can use for testing. Note the parser will let you put parentheses around any expression or statement and requires parentheses in certain positions.

(c) Define a formal small-step operational semantics for this new language. Hints/requirements:
- The inference rules should have the form $L; H; s \rightarrow L'; H'; s'$.
- Use the syntax $L ::= [] | H::L$ for lists.
- Because the form of rules has changed, include “all the rules” (i.e, rules for assignment, conditionals, etc.)
- You need 3 rules total for `pushheap` and `popheap x`.
- You do not need to write the semantics for expressions because they have not changed.

(d) In a short English paragraph, explain why our language would be much less useful if popping a heap did not copy one value from the current heap.

3. We will extend IMP with the ability to push and pop variables:

- A heap now maps a variable to a number and a list (i.e., a stack) of numbers. The initial heap maps each variable to 0 and the empty-list (though it’s not implemented that way).
- There are two new statement forms: `pushvar x` and `popvar x`. `pushvar x` changes $H$ such that $x$ maps to the same number and a list that is like the old list except the number is at the beginning. `popvar x` is the inverse of `pushvar x`, i.e., it changes $H$ such that $x$ maps to the number at the beginning of the list and that number is then removed from the beginning of the list. If $L$ has no effect if the list for $x$ is empty. Both `pushvar x` and `popvar x` “become skip” in one step. Example: Given a heap $H$ that contains $\{(x1,1,[6;3])),(y1,-3,[2])\}$, applying `pushvar x1` will return a heap $H'$ that contains $\{(x1,1,[1;6;3]),(y1,-3,[2])\}$ and `popvar x1` on $H'$ will return a heap that contains $\{(x1,1,[6;3]),(y1,-3,[2])\}$.

(a) Extend the provided interpreter to support this new language. Hints/requirements:
- Do not use mutation.
- Change only `get_var`, `set_var`, and `interp_step`, but also add functions `push_var` and `pop_var`, which take a heap and a string and return a heap. For `interp_step` you only need to add new cases. For `push_var` and `pop_var`, you typically produce a new heap with a different entry for the string. However, you must also do the right thing if there is no entry for the string, which is equivalent to there being an entry like `(str,0,[1])` “before the change".
• Just “add on” to your solution the previous problem. This addition is completely orthogonal.

(b) Give an IMP program using \texttt{pushvar} \(x\) and \texttt{popvar} \(x\) that we can use for testing.

(c) \textbf{Challenge Problem:} Define a formal small-step operational semantics for this new language. Be sure to give a precise definition for heaps, explaining your notation.
4. We will prove a property of execution for IMP with pushheap and popheap $x$.

(a) Formally prove the following: (For all $s$, $L$, $H$, and $s'$.) If $s$ has no while loops and $[] : s \rightarrow^* L; H; s'$ (i.e., we run $s$ for any number of steps), then the length of $L$ does not exceed the number of occurrences of pushheap in $s$ (i.e., in the original program). Hints:

- Use induction on the number of steps taken, and a second inductive proof on the height of the derivation of taking one step.
- For both proofs, you need to strengthen the induction hypothesis. Think through how to do so before writing out a long incorrect proof.

(b) Formally prove the following: The previous claim is false if we allow $s$ to contain while loops. Hint: Do not use induction.

5. Challenge Problem: Formally state and prove the following for our original IMP language: For any $s$ that does not contain the variable $x$, the IMP programs $x := x + 1; y := x \times x; s; x := x - 1$ and $y := (x + 1) \times (x + 1); s$ are equivalent.

What to turn in:

- OCaml source code for problem 1 in a file called `trees.ml`.
- Hard-copy (written or typed) answers to problems 1f, 2c, 2d, and 4.
- OCaml source code for problems 2a and 3a in a file called `interp.ml`.
- IMP code for problem 2b and 3b in files called `2b.imp` and `3b.imp`.
- For the challenge problems, edit `trees.ml` and your hardcopy solutions.

Do not modify interpreter files other than `interp.ml`. 