1 Amortized analysis: The accounting method

Let us use push() and multi-pop() operations for stacks as an example. The following is a sample of formal written arguments and also what you are expected to do in homework. Remember, we always start with an empty data-structure (e.g., an empty stack in this context).

Assign the deposit amount to each operation:
• 2$ for each push() operation.
• 0$ for each multi-pop() operation.

The key is then to prove that the bank balance remains non-negative during a series of push() and multi-pop() operations. There are multiple ways that one can argue about this claim. One particular way is to establish the following invariant during the execution of a series of operations.

# (bank) credits = # items in the stack

For our purpose, it suffices to prove the $\geq$ direction. It happens to be tight so that we have $\doteq$. One can also phrase the above invariant in terms of proof by inductions. The induction hypothesis could be

After the $i$th operation, # (bank) credits $\doteq$ # items in the stack.

Note that the above invariant (or induction hypothesis) implies that the balance is non-negative after each operation because the number of items in the stack is non-negative.

Let us continue with the induction language which is strict in mathematics. First, we argue about the base case $i = 0$, where no operation has been performed. The bank starts with 0 balance and the stack is empty. Thus, the induction hypothesis holds.

Now, we know after the $i$th operation, the bank balance $c$ equals the number of items $k$ in the stack. We want to establish the induction hypothesis after the $(i+1)$th operation. If the $(i+1)$th operation is

• push(), then we deposit 2$ and spend one for the push operation. The new bank balance $c' = c + 1$ and the new number of items $k' = k + 1$. Thus $c' = k'$ after the $(i+1)$th operation.

• multi-pop, then we deposit 0$ and spend $k$ for the multi-pop operation. The new bank balance $c' = c - k$ and the new number of items $k' = k - k$. Because $c = k$, we have $c' = k'$ after the $(i+1)$th operation.

Thus, we complete the proof by induction. The induction hypothesis implies that the bank balance is non-negative so that we are done. The amortized complexity of push() and multi-pop() is hence $O(1)$.

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1Contents will appear in homework and exams.