Assignment 1

You can either submit it electronically to Canvas or bring a physical copy to the lecture. This assignment is 12% in your total points. For the simplicity of the grading, the total points for the assignment is 60.

1 Problems that you don’t need to submit solutions

However, you probably want to figure out answers to these problems as well.

Problem 1.1. For complex number $c = a + bi$, recall that the real and imaginary parts of $c$ are denoted $\text{Re}(c) = a$ and $\text{Im}(c) = b$.

- Prove that $c + c^* = 2 \cdot \text{Re}(c)$.
- Prove that $cc^* = a^2 + b^2$.
- What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos \theta + i \sin \theta$?
- Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ as a vector in the complex plane.

Problem 1.2. Define that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- What is $\text{tr}(A |1\rangle \langle 0|)$? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of $A$. Do master this trick; it will be used repeatedly in the course and save you much time.)
- Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the same trick above, along with the fact that the trace is linear, to quickly evaluate $\text{tr}(A \cdot |+\rangle \langle +|)$.

Problem 1.3.

- Write out the 4-dimensional vector for $(\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle)$?
- Let $B_1 = \{|\psi_1\rangle, |\psi_2\rangle\}$, $B_2 = \{|\phi_1\rangle, |\phi_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^2$. Can you construct an orthonormal basis for $\mathbb{C}^4$?

Problem 1.4.

- Write out the $4 \times 4$ matrix representing $Y \otimes Y$.
- Prove that $(Z \otimes Y)^\dagger = Z \otimes Y$. Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and $Y$. 
2 Problems that you’re required to submit solutions

Please only submit the solutions to problems in this section.

Problem 2.1 [10 points]. Prove that for any normalized vectors $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$, 

$$\| |\psi\rangle - |\phi\rangle \|_2 = \sqrt{2 - 2 \cdot \Re(\langle \psi | \phi \rangle)}.$$ 

Why does it not matter if we replace $\langle \psi | \phi \rangle$ with $\langle \phi | \psi \rangle$ in this equation?

Problem 2.2 [10 points]. Use the spectral decompositions of $X$ and $Z$ to prove that $HXH^\dagger = Z$. (Do not simply write out the matrices and multiply!) Why does this immediately also yield that $HZH^\dagger = X$?

Problem 2.3 [20 points: 5 points each]. Distinguishing between pairs of quantum states. In each case, one of the two given states is randomly selected (probability 1/2 each) and given to you. You are not told which one it is. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

- $|0\rangle$ and $|+\rangle$. Recall that $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- $|0\rangle |0\rangle$ and $|+\rangle |+\rangle$ (note that this is like part (a), except you provided with two copies of the state)
- $|0\rangle |0\rangle$ and $\frac{1}{\sqrt{2}} |0\rangle |0\rangle + \frac{1}{\sqrt{2}} |1\rangle |1\rangle$
- $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta} |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle + e^{-i\theta} |1\rangle)$ (where $\theta \in [0, \pi/2]$ is known to you and your answer should be a function of $\theta$)

Problem 2.4 [20 points: 10 points each]. Distinguishing between identical and orthogonal states. Here we consider the problem where you are given two qubits as input that are either in the same state or their states are orthogonal to each other and the goal is to determine whether they are the same or not. We consider processes of this form: apply some 2-qubit unitary operation $U$ to the 2-qubit system and then measure the first qubit in the computational basis. If the two states are identical then outcome should be 0. If the two states are orthogonal then the outcome should be 1. If the two states are orthogonal then the outcome should be 1.

- Describe a unitary $U$ that perfectly distinguishes (in the above sense) in the special case where the input qubits are in computational basis states. That is, the outcome should be 0 for $|0\rangle |0\rangle$ and $|1\rangle |1\rangle$ and the outcome should be 1 for $|0\rangle |1\rangle$ and $|1\rangle |0\rangle$.

- Prove that, for all unitary operations $U$ that perfectly distinguish for computational basis states (such as the one in part (a)), they cannot also perfectly distinguish for states of the form $|+\rangle |+\rangle$, $|+\rangle |-\rangle$, $|-\rangle |+\rangle$, and $|-\rangle |-\rangle$. What is the maximum success probability possible for these states? (Recall that $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.)