Where we are

- Done: OCaml tutorial, “IMP” syntax, structural induction
- Now: Operational semantics for our little “IMP” language
  - Most of what you need for Homework 1
  - (But Problem 4 requires proofs over semantics)
IMP’s abstract syntax is defined inductively:

\[
\begin{align*}
  s &::= \text{skip} \mid x := e \mid s; s \mid \text{if } e \; s \; s \mid \text{while } e \; s \\
  e &::= c \mid x \mid e + e \mid e \ast e \\
  (c &\in \{\ldots, -2, -1, 0, 1, 2, \ldots \}) \\
  (x &\in \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots \})
\end{align*}
\]

We haven’t yet said what programs mean! (Syntax is boring)

Encode our “social understanding” about variables and control flow
Outline

Semantics for expressions

1. Informal idea; the need for heaps
2. Definition of heaps
3. The evaluation judgment (a relation form)
4. The evaluation inference rules (the relation definition)
5. Using inference rules
   - Derivation trees as interpreters
   - Or as proofs about expressions
6. Metatheory: Proofs about the semantics

Then semantics for statements

...
Informal idea

Given $e$, what $c$ does $e$ evaluate to?

$$1 + 2 \quad x + 2$$

It depends on the values of variables (of course)

Use a heap $H$ for a total function from variables to constants

- Could use partial functions, but then $\exists H$ and $e$ for which there is no $c$

We’ll define a relation over triples of $H$, $e$, and $c$

- Will turn out to be function if we view $H$ and $e$ as inputs and $c$ as output
- With our metalanguage, easier to define a relation and then prove it is a function (if, in fact, it is)
Heaps

\[ H ::= \cdot \mid H, x \mapsto c \]

A lookup-function for heaps:

\[
H(x) = \begin{cases} 
  c & \text{if } H = H', x \mapsto c \\
  H'(x) & \text{if } H = H', y \mapsto c' \text{ and } y \neq x \\
  0 & \text{if } H = \cdot 
\end{cases}
\]

- Last case avoids “errors” (makes function total)

“What heap to use” will arise in the semantics of statements

- For expression evaluation, “we are given an H”
The judgment

We will write: $H ; e \Downarrow c$

to mean, “$e$ evaluates to $c$ under heap $H$”

It is just a relation on triples of the form $(H, e, c)$

We just made up metasyntax $H ; e \Downarrow c$ to follow PL convention and to distinguish it from other relations

We can write: $., x \mapsto 3 ; x + y \Downarrow 3$, which will turn out to be true

(this triple will be in the relation we define)

Or: $., x \mapsto 3 ; x + y \Downarrow 6$, which will turn out to be false

(this triple will not be in the relation we define)
Inference rules

**CONST**

\[
H ; c \Downarrow c
\]

**VAR**

\[
H ; x \Downarrow H(x)
\]

**ADD**

\[
H ; e_1 \Downarrow c_1 \quad H ; e_2 \Downarrow c_2
H ; e_1 + e_2 \Downarrow c_1 + c_2
\]

**MULT**

\[
H ; e_1 \Downarrow c_1 \quad H ; e_2 \Downarrow c_2
H ; e_1 * e_2 \Downarrow c_1 * c_2
\]

Top: *hypotheses*
Bottom: *conclusion* (read first)

By definition, if all hypotheses hold, then the conclusion holds

Each rule is a *schema* you “instantiate consistently”

- So rules “work” “for all” \(H, c, e_1\), etc.
- But “each” \(e_1\) has to be the “same” expression
Instantiating rules

Example instantiation:

\[
\begin{align*}
\text{\cdot, } y & \mapsto 4 \ ; \ 3 + y \downarrow 7 \\
\text{\cdot, } y & \mapsto 4 \ ; \ 5 \downarrow 5 \\
\text{\cdot, } y & \mapsto 4 \ ; \ (3 + y) + 5 \downarrow 12
\end{align*}
\]

Instantiates:

\[
\text{ADD} \\
H \ ; e_1 \downarrow c_1 \quad H \ ; e_2 \downarrow c_2 \\
H \ ; e_1 + e_2 \downarrow c_1 + c_2
\]

with

\[
H = \cdot, \ y \mapsto 4 \\
e_1 = (3 + y) \\
c_1 = 7 \\
e_2 = 5 \\
c_2 = 5
\]
A \textit{(complete) derivation} is a tree of instantiations with \textit{axioms} at the leaves.

Example:

\[
\begin{align*}
\text{\cdot, } y &\mapsto 4 ; 3 \Downarrow 3 \quad &\text{\cdot, } y &\mapsto 4 ; y \Downarrow 4 \\
\text{\cdot, } y &\mapsto 4 ; 3 + y \Downarrow 7 \quad &\text{\cdot, } y &\mapsto 4 ; 5 \Downarrow 5 \\
\text{\cdot, } y &\mapsto 4 ; (3 + y) + 5 \Downarrow 12
\end{align*}
\]

By definition, \( H ; e \Downarrow c \) if there exists a derivation with \( H ; e \Downarrow c \) at the root.
Back to relations

So what relation do our inference rules define?

- Start with empty relation (no triples) \( R_0 \)

- Let \( R_i \) be \( R_{i-1} \) union all \( H ; e \downarrow c \) such that we can instantiate some inference rule to have conclusion \( H ; e \downarrow c \) and all hypotheses in \( R_{i-1} \)
  - So \( R_i \) is all triples at the bottom of height-\( j \) complete derivations for \( j \leq i \)

- \( R_\infty \) is the relation we defined
  - All triples at the bottom of complete derivations

For the math folks: \( R_\infty \) is the smallest relation closed under the inference rules
What are these things?

We can view the inference rules as defining an *interpreter*

- Complete derivation shows recursive calls to the “evaluate expression” function
  - Recursive calls from conclusion to hypotheses
  - *Syntax-directed* means the interpreter need not “search”

- See OCaml code in Homework 1

Or we can view the inference rules as defining a *proof system*

- Complete derivation proves facts from other facts starting with axioms
  - Facts established from hypotheses to conclusions
Some theorems

- **Progress**: For all $H$ and $e$, there exists a $c$ such that $H ; e \downarrow c$

- **Determinacy**: For all $H$ and $e$, there is at most one $c$ such that $H ; e \downarrow c$

We rigged it that way...

what would division, undefined-variables, or `gettime()` do?

Proofs are by induction on the the structure (i.e., height) of the expression $e$
On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.
  ▶ If it terminates

We could define $H_1 ; s \downarrow H_2$
  ▶ Would be a partial function from $H_1$ and $s$ to $H_2$
  ▶ Works fine; could be a homework problem

Instead we’ll define a “small-step” semantics and then “iterate” to “run the program”

\[
H_1 ; s_1 \rightarrow H_2 ; s_2
\]
Statement semantics

\[
H_1 \ ;\ s_1 \rightarrow H_2 \ ;\ s_2
\]

**ASSIGN**

\[
\frac{H \ ;\ e \downarrow c}{H \ ;\ x := e \rightarrow H, x \leftrightarrow c \ ;\ skip}
\]

**SEQ1**

\[
\frac{H \ ;\ skip;\ s \rightarrow H \ ;\ s}{H \ ;\ s}
\]

**SEQ2**

\[
\frac{H \ ;\ s_1 \rightarrow H' \ ;\ s_1'}{H \ ;\ s_1;\ s_2 \rightarrow H' \ ;\ s_1';\ s_2}
\]

**IF1**

\[
\frac{H \ ;\ e \downarrow c \quad c > 0}{H \ ;\ if\ e\ s_1\ s_2 \rightarrow H \ ;\ s_1}
\]

**IF2**

\[
\frac{H \ ;\ e \downarrow c \quad c \leq 0}{H \ ;\ if\ e\ s_1\ s_2 \rightarrow H \ ;\ s_2}
\]
What about \textbf{while} $e \; s$ (do $s$ and loop if $e > 0$)?

\[
\text{WHILE} \\
H \; \text{while} \; e \; s \rightarrow H \; \text{if} \; e \; (s; \text{while} \; e \; s) \; \text{skip}
\]

Many other equivalent definitions possible
Program semantics

Defined $H ; s \rightarrow H' ; s'$, but what does “s” mean/do?

Our machine iterates: $H_1 ; s_1 \rightarrow H_2 ; s_2 \rightarrow H_3 ; s_3 \ldots$, 
with each step justified by a complete derivation using our single-step statement semantics

Let $H_1 ; s_1 \rightarrow^n H_2 ; s_2$ mean “becomes after n steps”

Let $H_1 ; s_1 \rightarrow^* H_2 ; s_2$ mean “becomes after 0 or more steps”

Pick a special “answer” variable ans

The program s produces c if · ; s →^* H ; skip and $H(\text{ans}) = c$

Does every s produce a c?
Example program execution

\[ x := 3; (y := 1; \textbf{while} \ x \ (y := y \times x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \times x; x := x - 1) \).

\[
\begin{align*}
\cdot; x := 3; y := 1; \textbf{while} \ x \ s \\
\rightarrow \quad \cdot; x \mapsto 3; \textbf{skip}; y := 1; \textbf{while} \ x \ s \\
\rightarrow \quad \cdot; x \mapsto 3; y \mapsto 1; \textbf{while} \ x \ s \\
\rightarrow^2 \quad \cdot; x \mapsto 3, y \mapsto 1; \textbf{while} \ x \ s \\
\rightarrow \quad \cdot; x \mapsto 3, y \mapsto 1; \textbf{if} \ x \ (s; \textbf{while} \ x \ s) \ \textbf{skip} \\
\rightarrow \quad \cdot; x \mapsto 3, y \mapsto 1; y := y \times x; x := x - 1; \textbf{while} \ x \ s
\end{align*}
\]
Continued...

\[ \rightarrow^2 \text{ . } x \mapsto 3, y \mapsto 1, y \mapsto 3; \ x := x - 1; \text{ while } x \ s \]

\[ \rightarrow^2 \text{ . } x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \text{ while } x \ s \]

\[ \rightarrow \text{ . . . , } y \mapsto 3, x \mapsto 2; \text{ if } x (s; \text{ while } x \ s) \text{ skip} \]

\[ \text{ . . . } \]

\[ \rightarrow \text{ . . . , } y \mapsto 6, x \mapsto 0; \text{ skip} \]
Where we are

Defined $H; e \downarrow c$ and $H; s \rightarrow H'; s'$ and extended the latter to give $s$ a meaning

- The way we did expressions is “large-step operational semantics”
- The way we did statements is “small-step operational semantics”
- So now you have seen both

Definition by interpretation: program means what an interpreter (written in a metalanguage) says it means

- Interpreter represents a (very) abstract machine that runs code

Large-step does not distinguish errors and divergence

- But we defined IMP to have no errors
- And expressions never diverge
Establishing Properties

We can prove a property of a terminating program by “running” it.

Example: Our last program terminates with \( x \) holding \( 0 \).

We can prove a program diverges, i.e., for all \( H \) and \( n \),
\[
\cdot; s \rightarrow^n H ; \text{skip}
\]
cannot be derived.

Example: \textbf{while 1 skip}

By induction on \( n \), but requires a \textit{stronger induction hypothesis}.
More General Proofs

We can prove properties of executing all programs (satisfying another property)

Example: If $H$ and $s$ have no negative constants and $H ; s \rightarrow^* H' ; s'$, then $H'$ and $s'$ have no negative constants.

Example: If for all $H$, we know $s_1$ and $s_2$ terminate, then for all $H$, we know $H ; (s_1 ; s_2)$ terminates.