Graduate Programming Languages:
Type Safety for STLC with Constants

Most of this is available in the slides. However, it can help to see it all in one place.

Syntax

\[ e ::= c \mid \lambda x.\ e \mid x \mid e\ e \]
\[ v ::= c \mid \lambda x.\ e \]
\[ \tau ::= \text{int} \mid \tau \rightarrow \tau \]
\[ \Gamma ::= \cdot \mid \Gamma, x : \tau \]

Evaluation Rules (a.k.a. Dynamic Semantics)

\[ e \rightarrow e' \]

\begin{align*}
\text{E-App} & : (\lambda x.\ e)\ v \rightarrow e[v/x] \\
\text{E-App1} & : e_1 \rightarrow e_1' \quad e_1\ e_2 \rightarrow e_1'\ e_2 \\
\text{E-App2} & : e_2 \rightarrow e_2' \quad v\ e_2 \rightarrow v\ e_2'
\end{align*}

Typing Rules (a.k.a. Static Semantics)

\[ \Gamma \vdash e : \tau \]

\begin{align*}
\text{T-Const} & : \Gamma \vdash c : \text{int} \\
\text{T-Var} & : \Gamma \vdash x : \Gamma(x) \\
\text{T-Fun} & : \Gamma, x : \tau_1 \vdash e : \tau_2 \quad x \notin \text{Dom}(\Gamma) \\
\text{T-App} & : \Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash e_2 : \tau_2 \\
& \quad \Gamma \vdash e_1\ e_2 : \tau_1
\end{align*}
Type Soundness

Theorem (Type Soundness). If $\cdot \vdash e : \tau$ and $e \rightarrow^* e'$, then either $e'$ is a value or there exists an $e''$ such that $e' \rightarrow e''$.

Proof

The Type Soundness Theorem follows as a simple corollary to the Progress and Preservation Theorems stated and proven below: Given the Preservation Theorem, a trivial induction on the number of steps taken to reach $e'$ from $e$ establishes that $\cdot \vdash e' : \tau$. Then the Progress Theorem ensures $e'$ is a value or can step to some $e''$.

We need the following lemma for our proof of Progress, below.

Lemma (Canonical Forms). If $\cdot \vdash v : \tau$, then

i If $\tau$ is int, then $v$ is a constant, i.e., some $c$.

ii If $\tau$ is $\tau_1 \rightarrow \tau_2$, then $v$ is a lambda, i.e., $\lambda x. e$ for some $x$ and $e$.

Canonical Forms. The proof is by inspection of the typing rules.

i If $\tau$ is int, then the only rule which lets us give a value this type is T-Const.

ii If $\tau$ is $\tau_1 \rightarrow \tau_2$, then the only rule which lets us give a value this type is T-Fun.

Theorem (Progress). If $\cdot \vdash e : \tau$, then either $e$ is a value or there exists some $e'$ such that $e \rightarrow e'$.

Progress. The proof is by induction on (the height of) the derivation of $\cdot \vdash e : \tau$, proceeding by cases on the bottommost rule used in the derivation.

T-Const $e$ is a constant, which is a value, so we are done.

T-Var Impossible, as $\Gamma$ is $\cdot$.

T-Fun $e$ is $\lambda x. e'$, which is a value, so we are done.

T-App $e$ is $e_1 e_2$.

By inversion, $\cdot \vdash e_1 : \tau' \rightarrow \tau$ and $\cdot \vdash e_2 : \tau'$ for some $\tau'$. If $e_1$ is not a value, then $\cdot \vdash e_1 : \tau' \rightarrow \tau$ and the induction hypothesis ensures $e_1 \rightarrow e'_1$ for some $e'_1$. Therefore, by E-App1, $e_1 e_2 \rightarrow e'_1 e_2$.

Else $e_1$ is a value. If $e_2$ is not a value, then $\cdot \vdash e_2 : \tau'$ and our induction hypothesis ensures $e_2 \rightarrow e'_2$ for some $e'_2$. Therefore, by E-App2, $e_1 e_2 \rightarrow e_1 e'_2$.

Else $e_1$ and $e_2$ are values. Then $\cdot \vdash e_1 : \tau' \rightarrow \tau$ and the Canonical Forms Lemma ensures $e_1$ is some $\lambda x. e'$. And ($\lambda x. e'$) $e_2 \rightarrow e'[e_2/x]$ by E-Apply, so $e_1 e_2$ can take a step.
We will need the following lemma for our proof of Preservation, below. Actually, in the proof of Preservation, we need only a Substitution Lemma where \( \Gamma \) is \( \cdot \), but proving the Substitution Lemma itself requires the stronger induction hypothesis using any \( \Gamma \).

**Lemma (Substitution).** If \( \Gamma, x: \tau' \vdash e : \tau \) and \( \Gamma \vdash e' : \tau' \), then \( \Gamma \vdash e'[x]/x : \tau \).

To prove this lemma, we will need the following two technical lemmas, which we will assume without proof (they’re not that difficult).

**Lemma (Weakening).** If \( \Gamma \vdash e : \tau \) and \( x \notin \text{Dom}(\Gamma) \), then \( \Gamma, x: \tau' \vdash e : \tau \).

**Lemma (Exchange).** If \( \Gamma, x: \tau_1, y: \tau_2 \vdash e : \tau \) and \( y \neq x \), then \( \Gamma, y: \tau_2, x: \tau_1 \vdash e : \tau \).

Now we prove Substitution.

**Substitution.** The proof is by induction on the derivation of \( \Gamma, x: \tau' \vdash e : \tau \). There are four cases. In all cases, we know \( \Gamma \vdash e' : \tau' \) by assumption.

**T-Const** \( e \) is \( c \), so \( c[e'/x] \) is \( c \). By T-Const, \( \Gamma \vdash c : \text{int} \).

**T-Var** \( e \) is \( y \) and \( \Gamma, x: \tau' \vdash y : \tau \).

If \( y \neq x \), then \( y[e'/x] \) is \( y \). By inversion on the typing rule, we know that \( (\Gamma, x: \tau')(y) = \tau \). Since \( y \neq x \), we know that \( \Gamma(y) = \tau \). So by T-Var, \( \Gamma \vdash y : \tau \).

If \( y = x \), then \( y[e'/x] \) is \( e' \). \( \Gamma, x: \tau' \vdash x : \tau \), so by inversion, \( (\Gamma, x: \tau')(x) = \tau \), so \( \tau = \tau' \).

We know \( \Gamma \vdash e' : \tau' \), which is exactly what we need.

**T-App** \( e \) is \( e_1 e_2 \), so \( e[e'/x] \) is \( (e_1[e'/x])(e_2[e'/x]) \).

We know \( \Gamma, x: \tau' \vdash e_1 e_2 : \tau_1 \), so, by inversion on the typing rule, we know \( \Gamma, x: \tau' \vdash e_1 : \tau_2 \rightarrow \tau_1 \) and \( \Gamma, x: \tau' \vdash e_2 : \tau_2 \) for some \( \tau_2 \).

Therefore, by induction, \( \Gamma \vdash e_1[e'/x] : \tau_2 \rightarrow \tau_1 \) and \( \Gamma \vdash e_2[e'/x] : \tau_2 \).

Given these, T-App lets us derive \( \Gamma \vdash (e_1[e'/x])(e_2[e'/x]) : \tau_1 \).

So by the definition of substitution \( \Gamma \vdash (e_1 e_2)[e'/x] : \tau_1 \).

**T-Fun** \( e \) is \( \lambda y. \ e_b \), so \( e[e'/x] \) is \( \lambda y. \ (e_b[e'/x]) \).

We can \( \alpha \)-convert \( \lambda y. \ e_b \) to ensure \( y \notin \text{Dom}(\Gamma) \) and \( y \neq x \).

We know \( \Gamma, x: \tau' \vdash \lambda y. \ e_b : \tau_1 \rightarrow \tau_2 \), so, by inversion on the typing rule, we know \( \Gamma, x: \tau', y: \tau_1 \vdash e_b : \tau_2 \).

By Exchange, we know that \( \Gamma, y: \tau_1, x: \tau' \vdash e_b : \tau_2 \).

By Weakening, we know that \( \Gamma, y: \tau_1 \vdash e' : \tau' \).

We have rearranged the two typing judgments so that our induction hypothesis applies (using \( \Gamma, y: \tau_1 \) for the typing context called \( \Gamma \) in the statement of the lemma), so, by induction, \( \Gamma, y: \tau_1 \vdash e_b[e'/x] : \tau_2 \).

Given this, T-Fun lets us derive \( \Gamma \vdash \lambda y. \ e_b[e'/x] : \tau_1 \rightarrow \tau_2 \).

So by the definition of substitution, \( \Gamma \vdash (\lambda y. \ e_b)[e'/x] : \tau_1 \rightarrow \tau_2 \).
Theorem (Preservation). If \( \cdot \vdash e : \tau \) and \( e \rightarrow e' \), then \( \cdot \vdash e' : \tau \).

Preservation. The proof is by induction on the derivation of \( \cdot \vdash e : \tau \). There are four cases.

T-Const \( e \) is \( c \). This case is impossible, as there is no \( e' \) such that \( c \rightarrow e' \).

T-Var \( e \) is \( x \). This case is impossible, as \( x \) cannot be typechecked under the empty context.

T-Fun \( e \) is \( \lambda x. e_b \). This case is impossible, as there is no \( e' \) such that \( \lambda x. e_b \rightarrow e' \).

T-App \( e \) is \( e_1 e_2 \), so \( \cdot \vdash e_1 e_2 : \tau \).

By inversion on the typing rule, \( \cdot \vdash e_1 : \tau_2 \rightarrow \tau \) and \( \cdot \vdash e_2 : \tau_2 \) for some \( \tau_2 \).

There are three possible rules for deriving \( e_1 e_2 \rightarrow e' \).

E-App1 Then \( e' = e_1' e_2 \) and \( e_1 \rightarrow e_1' \).

By \( \cdot \vdash e_1 : \tau_2 \rightarrow \tau \), \( e_1 \rightarrow e_1' \), and induction, \( \cdot \vdash e_1' : \tau_2 \rightarrow \tau \).

Using this and \( \cdot \vdash e_2 : \tau_2 \), T-App lets us derive \( \cdot \vdash e_1' e_2 : \tau \).

E-App2 Then \( e' = e_1 e_2' \) and \( e_2 \rightarrow e_2' \).

By \( \cdot \vdash e_2 : \tau_2 \), \( e_2 \rightarrow e_2' \), and induction \( \cdot \vdash e_2' : \tau_2 \).

Using this and \( \cdot \vdash e_1 : \tau_2 \rightarrow \tau \), T-App lets us derive \( \cdot \vdash e_1 e_2' : \tau \).

E-Apply Then \( e_1 \) is \( \lambda x. e_b \) for some \( x \) and \( e_b, \) and \( e' = e_b[e_2/x] \).

By inversion of the typing of \( \cdot \vdash e_1 : \tau_2 \rightarrow \tau \), we have \( \cdot, x: \tau_2 \vdash e_b : \tau \).

This and \( \cdot \vdash e_2 : \tau_2 \) lets us use the Substitution Lemma to conclude \( \cdot \vdash e_b[e_2/x] : \tau \).

\( \square \)