x. Greedy Loop Invariant – extra point

An element \( x \) is the majority element in a multiset \( S \) if its multiplicity exceeds that of all the other elements combined.

(i) Prove that deleting two unequal elements from \( S \) preserves the identity of the majority element, if one exists: Assume \( |S| = n \) and multiplicity of \( x, m_S(x) > n/2 \). There are two cases of deleting two unequal elements of \( S \).

1. None of the deleted elements is \( x \). Then the multiplicity of \( x \) in \( S' = S \setminus \{y, z\} \) is \( m_{S'} = m_S(x) > n/2 > (n - 2)/2 = |S'|/2 \), ie., \( x \) remains a majority element of \( S' \).

2. One of the deleted elements is \( x \). Then the multiplicity of \( x \) in \( S' = S \setminus \{y, x\} \) is \( m_{S'} = m_S(x) - 1 > n/2 - 1 = (n - 2)/2 = |S'|/2 \), ie., \( x \) remains a majority element of \( S' \).

(ii) Based on this invariant, provide a linear time algorithm to determine the majority element in a given multiset, if such a majority exists.

Such an algorithm is allowed only a constant number of examinations per element of the given set. Maintaining the lower bound on the multiplicity \( m > 0 \) of the purported majority element so far, candidate, the algorithm compares candidate with the next element. If they are identical, \( m \) is incremented, otherwise it is decremented (as both elements are deleted). If \( m = 0 \) (true initially) candidate takes the identity of next and \( m \) is incremented. When next no longer exists (unequal pairs have been deleted), only the candidate can be the majority element, which can be checked by an additional scan of the set.