Estimation of Absolute Performance

- Variability in outputs measures
- Terminating –vs- Steady State simulations
- Point estimators for terminating simulations
- Confidence intervals for terminating simulations
- Warm-up time and how to treat steady state simulations
Variability in Output Measures

Single Server Queue
Arrival Rate ~ Exp
Service Rate ~ Norm

Performance Measures:
• Average queue size
• Max queue size
• Average waiting time
• Max waiting time
• Average time in system
• Max time in system
• Server utilization
• Number served

• Made a total of five runs of the simulation.
• Total Production on each run was:
  - 5
  - 3
  - 6
  - 2
  - 3

So which value for Production is correct?
Variability in Output Measures

• Made a total of five replications:

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Replication</th>
<th>Sample</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3 4 5</td>
<td>Avg.</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td>Total production</td>
<td>5 3 6 2 3</td>
<td>3.80</td>
<td>1.64</td>
</tr>
<tr>
<td>Average waiting time in queue</td>
<td>2.53 1.19 1.03 1.62 0.00</td>
<td>1.27</td>
<td>0.92</td>
</tr>
<tr>
<td>Maximum waiting time in queue</td>
<td>8.16 3.56 2.97 3.24 0.00</td>
<td>3.59*</td>
<td>2.93*</td>
</tr>
<tr>
<td>Average total time in system</td>
<td>6.44 5.16 4.16 6.71 4.26</td>
<td>5.33</td>
<td>1.19</td>
</tr>
<tr>
<td>Maximum total time in system</td>
<td>12.62 6.63 6.27 7.71 4.96</td>
<td>7.64*</td>
<td>2.95*</td>
</tr>
<tr>
<td>Time-average number of parts in queue</td>
<td>0.79 0.18 0.36 0.16 0.05</td>
<td>0.31</td>
<td>0.29</td>
</tr>
<tr>
<td>Maximum number of parts in queue</td>
<td>3 1 2 1 1</td>
<td>1.60*</td>
<td>0.89*</td>
</tr>
<tr>
<td>Drill-press utilization</td>
<td>0.92 0.59 0.90 0.51 0.70</td>
<td>0.72</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Output Measures | Absolute performance measures

Terminating vs Steady State Sim.

• Terminating Simulation
  - Distinct start and stop times for process
  - Tends to start with no entities in system and end with no entities
  Examples:
  - Fast food
  - Bassett furniture door frame assembly cell
  - Evaluating manpower loading requirements for depot maintenance activity network.

• Steady State Simulation
  - Process tends to run “forever”
  - Entities stay in the system between shift changes
  - Interested in long-term trends
  Examples:
  - Assembly lines (Ford)
  - Evaluating WIP in depot maintenance
  - Warehouse operations (Double eagle steel)
  - Nashville glass floatline

Output Analysis treated somewhat differently.
### Point Estimation – Terminating Sim.

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Replication</th>
<th>Sample Avg</th>
<th>Sample Std. Dev</th>
<th>95% Half Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total production</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Average waiting time in queue</td>
<td>2.53</td>
<td>1.19</td>
<td>1.03</td>
<td>1.62</td>
</tr>
<tr>
<td>Maximum waiting time in queue</td>
<td>8.16</td>
<td>3.56</td>
<td>2.97</td>
<td>3.24</td>
</tr>
<tr>
<td>Average total time in system</td>
<td>6.44</td>
<td>5.10</td>
<td>4.16</td>
<td>6.71</td>
</tr>
<tr>
<td>Maximum total time in system</td>
<td>12.62</td>
<td>6.63</td>
<td>6.27</td>
<td>7.71</td>
</tr>
<tr>
<td>Time-average number of parts in queue</td>
<td>0.79</td>
<td>0.18</td>
<td>0.26</td>
<td>0.16</td>
</tr>
<tr>
<td>Maximum number of parts in queue</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Drill-press utilization</td>
<td>0.92</td>
<td>0.59</td>
<td>0.90</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Statistical mean or average

\[
\bar{\theta} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \text{or} \quad \bar{T} = \frac{1}{T_e} \int_{0}^{T_e} Y(t) dt
\]

<table>
<thead>
<tr>
<th>Tally</th>
<th>Time-Persistent</th>
</tr>
</thead>
</table>

Which of the above performance measures are time-persistent and which are tally variables?
Absolute Measure of Performance Estimation

• Made a total of five replications:

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Replication</th>
<th>Sample Avg</th>
<th>Std. Dev</th>
<th>Half Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total production</td>
<td>5 3 6 2 3</td>
<td>3.80</td>
<td>1.64</td>
<td>2.04</td>
</tr>
<tr>
<td>Average waiting time in queue</td>
<td>2.53 1.19 1.03 1.62 0.00</td>
<td>1.27</td>
<td>0.92</td>
<td>1.14</td>
</tr>
<tr>
<td>Maximum waiting time in queue</td>
<td>8.16 3.56 2.97 3.74 0.00</td>
<td>3.59*</td>
<td>2.93*</td>
<td>3.65*</td>
</tr>
<tr>
<td>Average time in system</td>
<td>6.44 5.10 4.14 6.71 4.26</td>
<td>5.33</td>
<td>1.19</td>
<td>1.48</td>
</tr>
<tr>
<td>Maximum time in system</td>
<td>12.62 6.63 6.27 7.71 4.96</td>
<td>7.64*</td>
<td>2.95*</td>
<td>3.67*</td>
</tr>
<tr>
<td>Time-average number of parts in queue</td>
<td>0.79 0.18 0.36 0.16 0.05</td>
<td>0.31</td>
<td>0.29</td>
<td>0.36</td>
</tr>
<tr>
<td>Maximum number of parts in queue</td>
<td>1 1 2 1 1</td>
<td>1.60</td>
<td>0.89*</td>
<td>1.11*</td>
</tr>
<tr>
<td>Drill-press utilization</td>
<td>0.92 0.59 0.90 0.51 0.70</td>
<td>0.72</td>
<td>0.18</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Sample mean or average

\[ \hat{\theta} = \bar{Y} = \frac{1}{R} \sum_{i=1}^{R} Y_i \]

Sample variance (\( S^2 \)) or standard deviation (S)

\[ S^2 = \frac{1}{R-1} \sum_{i=1}^{R} \left( Y_i - \bar{Y} \right)^2 \]

Confidence Intervals

• A measure of error
• The sample averages (Theta and Phi) and standard deviation (S) are estimates
• CI attempts to bound the error
• A confidence level (e.g. \( 95\% = 1-\alpha \)) tells how much we can trust the true mean to be within the CI

\[ CI = \bar{Y} \pm t_{\alpha/2, R-1} \frac{S}{\sqrt{R}} \]
Confidence Intervals

• A CI can be somewhat controlled when performing a simulation analysis
• How can we tighten the CI?

\[ CI = \bar{Y} \pm t_{a/2, R-1} \frac{S}{\sqrt{R}} \]

• Make more replications / runs (increase R)
• Reduce S? How?
Absolute Measure of Performance Estimation

• **Made a total of five replications:**

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Replication</th>
<th>Sample Avg.</th>
<th>Std. Dev.</th>
<th>95% Half Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total production</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Average waiting time in queue</td>
<td>2.53</td>
<td>1.19</td>
<td>1.03</td>
<td>0.00</td>
</tr>
<tr>
<td>Maximum waiting time in queue</td>
<td>8.16</td>
<td>3.56</td>
<td>2.97</td>
<td>3.24</td>
</tr>
<tr>
<td>Average total time in system</td>
<td>6.44</td>
<td>5.10</td>
<td>4.16</td>
<td>4.26</td>
</tr>
<tr>
<td>Maximum total time in system</td>
<td>12.62</td>
<td>6.63</td>
<td>6.27</td>
<td>4.96</td>
</tr>
<tr>
<td>Time-average number of parts</td>
<td>0.79</td>
<td>0.18</td>
<td>0.36</td>
<td>0.16</td>
</tr>
<tr>
<td>Maximum number of parts in</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Drill-press utilization</td>
<td>0.92</td>
<td>0.59</td>
<td>0.90</td>
<td>0.51</td>
</tr>
</tbody>
</table>

95% Half Width

\[ \text{HalfWidth} = t_{\alpha/2,n-1} \frac{S}{\sqrt{R}} \]

**Prediction Intervals**

• Similar to CI
• Estimates/predicts a future outcome

\[ PI = \bar{Y} \pm t_{\alpha/2,n-1} S \sqrt{\frac{1}{n} + \frac{1}{n-1}} \]

• PI accounts for the natural variation in the process
• Cannot significantly “tighten” the interval by making more runs of the simulation
• Predicts the range of the output measure
Confidence vs Prediction Intervals

- Overall average of average cycle time on 120 replications of a simulation is 5.8 hrs, with a sample std. dev. of 1.6 hrs.
- \( t_{0.025, 119} = 1.98 \) (from table A.5.)
- A 95% CI=\(5.8 \pm 1.98(1.6/\sqrt{120})=5.8 \pm 0.29\)
- Our estimate is 5.8 hrs
- Error as large as \( \pm 0.29 \) hrs
- We are 95% confident that the average cycle time on that day will be \(\text{PI}=5.8 \pm 1.98(1.6)\sqrt{(1+1/120)}=5.8 \pm 3.18\)
- We are 95% confident of covering the actual average cycle on a particular day

Output Analysis for Terminating Simulations

<table>
<thead>
<tr>
<th>Within-Rep Data</th>
<th>Across-Rep Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_{11} ), ( Y_{12} ), \ldots, ( Y_{1n_1} )</td>
<td>( \bar{Y}_1, S^2_1, H_1 )</td>
</tr>
<tr>
<td>( Y_{21} ), ( Y_{22} ), \ldots, ( Y_{2n_2} )</td>
<td>( \bar{Y}_2, S^2_2, H_2 )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( Y_{R1} ), ( Y_{R2} ), \ldots, ( Y_{Rn_R} )</td>
<td>( \bar{Y}_R, S^2_R, H_R )</td>
</tr>
<tr>
<td>( \bar{Y}_\cdot, S^2 ), H</td>
<td>( \bar{Y}_\cdot, S^2 ), H</td>
</tr>
</tbody>
</table>

- \( \bar{Y}_i \) sample mean from the ith replication
- \( S^2_i \) sample variance
- \( H_i = t_{\alpha/2, n_i-1} \frac{S_i}{\sqrt{n_i}} \)
Output Analysis for Terminating Simulations

\[ \tilde{Y}_r = \frac{1}{R} \sum_{i=1}^{R} \tilde{Y}_i, \]

\[ S^2 = \frac{1}{R - 1} \sum_{i=1}^{R} (\tilde{Y}_i - \tilde{Y}_r)^2 \]

\[ H = t_{\alpha/2,R-1} \frac{S}{\sqrt{R}} \]

- \( S/\sqrt{R} \) = standard error - average error in \( \tilde{Y}_r \) as an estimator of \( \theta \)

### Output Analysis for Terminating Simulations

<table>
<thead>
<tr>
<th>Replication, ( r )</th>
<th>Average Waiting Time, ( \hat{w}_{Qr} ) (minutes)</th>
<th>Average on Hold, ( \hat{L}_{Qr} ) (people)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.88</td>
<td>0.68</td>
</tr>
<tr>
<td>2</td>
<td>5.04</td>
<td>4.18</td>
</tr>
<tr>
<td>3</td>
<td>4.13</td>
<td>3.26</td>
</tr>
<tr>
<td>4</td>
<td>0.52</td>
<td>0.34</td>
</tr>
</tbody>
</table>

\[ \bar{Y}_r = \hat{w}_Q = \frac{0.88 + 5.04 + 4.13 + 0.52}{4} = 2.64 \]

\[ S^2 = \frac{(0.88 - 2.64)^2 + \cdots + (0.52 - 2.64)^2}{4 - 1} = (2.28)^2 \]

\[ H = t_{0.025,3} \frac{S}{\sqrt{4}} = (3.18)(1.14) = 3.62 \]

2.64 ± 3.62 minutes with 95% confidence
Output Analysis for Terminating Simulations

CI – Specified Precision

\[ H = t_{\alpha/2, R-1} \frac{S}{\sqrt{R}} \]

- We don’t set R and accept the CI
- We set R large enough to produce H to a size to facilitate a decision supported by the simulation

- Estimate \( \theta \) by \( \bar{Y}_n \) to within \( \pm \epsilon \) with probability \( 1 - \alpha \)

\[
P(\mid \bar{Y}_n - \theta \mid \leq \epsilon) \geq 1 - \alpha
\]

\[ H = t_{\alpha/2, R-1} \frac{S_0}{\sqrt{R}} \leq \epsilon \]

\[ R \geq \left( \frac{t_{\alpha/2, R-1}S_0}{\epsilon} \right)^2 \]
CI – Specified Precision

• Since student’s t distribution needs R, we approximate by the normal distribution
  \[ R \geq \left( \frac{z_{\alpha/2} S_0}{\epsilon} \right)^2 \]

• \( z_{\alpha/2} \) is the 100(1-\( \alpha/2 \)) % of the standard normal distribution (Table A.3)

• Final CI
  \[ \bar{Y}_- - t_{\alpha/2, R-1} \frac{S}{\sqrt{R}} \leq \theta \leq \bar{Y}_+ + t_{\alpha/2, R-1} \frac{S}{\sqrt{R}} \]

Note:
• 1.995 obtained from student’s t distribution
• the example in the book is based on 20 runs, not 4

CI – Specified Precision

• Estimate mean waiting time in SMP’s hold queue to within ±0.5 min with 95% confidence

• \( R \geq (1.96 \times 2.28 / 0.5)^2 = 79.88 \)

• \( R = 80 \)

• \( \bar{Y}_- - 1.995 \frac{S}{\sqrt{80}} \leq \theta \leq \bar{Y}_+ - 1.995 \frac{S}{\sqrt{80}} \)

Note:
• 1.995 obtained from student’s t distribution
• the example in the book is based on 20 runs, not 4
Steady State Models – Warm Up and Run Length

• Most models start empty and idle
  § Empty: No entities are present at time 0
  § Idle: All resources are idle at time 0
  § In a terminating simulation this is OK if realistic
  § In a steady-state simulation, though, this can bias the output for a while after startup
    – Bias can go either way
    – Usually downward (results are biased low) in queueing-type models that eventually get congested
    – Depending on model, parameters, and run length, the bias can be very severe

Steady State Models – Warm Up and Run Length

• Remedies for initialization bias
  § Better starting state, more typical of steady state
    – Throw some entities around the model
    – Can be inconvenient to do this in the model
    – How do you know how many to throw and where? This is what you’re trying to estimate in the first place!
  § Make the run so long that bias is overwhelmed
    – Might work if initial bias is weak or dissipates quickly
  § Let model warm up, still starting empty and idle
    – Run > Setup > Replication Parameters: Warm-up Period
    – “Clears” all statistics at that point for summary report, any Outputs-type saved data from Statistic module of results across replications
Steady State Models – Warm Up and Run Length

Specified initial conditions, \[ t_0 \]

“Steady-state” initial conditions, \[ I \]

\[ 0 \quad T_0 \quad T_0 + T_E \]

Initialization phase of length \( T_0 \)

Data-collection phase of length \( T_E \)

---

Steady State Models – Warm Up and Run Length

<table>
<thead>
<tr>
<th>Replication</th>
<th>Observations</th>
<th>Replication Averages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Y_{1,1} ) ( \cdots ) ( Y_{1,d} ) ( Y_{1,d+1} ) ( \cdots ) ( Y_{1,n} )</td>
<td>( \bar{Y}_{1}(n, d) )</td>
</tr>
<tr>
<td>2</td>
<td>( Y_{2,1} ) ( \cdots ) ( Y_{2,d} ) ( Y_{2,d+1} ) ( \cdots ) ( Y_{2,n} )</td>
<td>( \bar{Y}_{2}(n, d) )</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>( R )</td>
<td>( Y_{R,1} ) ( \cdots ) ( Y_{R,d} ) ( Y_{R,d+1} ) ( \cdots ) ( Y_{R,n} )</td>
<td>( \bar{Y}_{R}(n, d) )</td>
</tr>
<tr>
<td>( \tilde{Y}_1 ) ( \cdots ) ( \tilde{Y}<em>d ) ( \tilde{Y}</em>{d+1} ) ( \cdots ) ( \tilde{Y}_n )</td>
<td>( \bar{Y}) ( (n, d) )</td>
<td></td>
</tr>
</tbody>
</table>
### Steady State Models – Warm Up and Run Length

\[
\bar{Y}_r(n, d) = \frac{1}{n-d} \sum_{j=d+1}^{n} Y_{rj}
\]

\[
\bar{Y}_1(n, d), \ldots, \bar{Y}_R(n, d)
\]

\[
\theta_{n,d} = \mathbb{E}[\bar{Y}_r(n, d)]
\]

\[
\bar{Y}_.(n, d) = \frac{1}{R} \sum_{r=1}^{R} \bar{Y}_r(n, d)
\]

\[
\mathbb{E}[\bar{Y}_.(n, d)] = \theta_{n,d}
\]

---

### Steady State Models – Warm Up and Run Length

\[
S^2 = \frac{1}{R-1} \sum_{r=1}^{R} (\bar{Y}_r - \bar{Y}_.)^2 = \frac{1}{R-1} \left( \sum_{r=1}^{R} \bar{Y}_r^2 - R \bar{Y}_.`^2 \right)
\]

\[
\text{s.e.}(\bar{Y}_.) = \frac{S}{\sqrt{R}}
\]

\[
\bar{Y}_. - t_{\alpha/2,R-1} \frac{S}{\sqrt{R}} \leq \theta \leq \bar{Y}_. + t_{\alpha/2,R-1} \frac{S}{\sqrt{R}}
\]
Example – Steady State Models

• Fast Chip Wafer Simulation
• R=10 replications, T=2200 hrs
• T₀=200, Tₑ=2000
• 60% of the production was type C chips
• 2000 x 0.6 = 1200 cycles per replication
• What is the long-run average cycle time?
• What is the error of the estimate at 95% confidence level?

<table>
<thead>
<tr>
<th>Replication r</th>
<th>Sample Mean for Replication r</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>46.86</td>
</tr>
<tr>
<td>2</td>
<td>46.09</td>
</tr>
<tr>
<td>3</td>
<td>47.64</td>
</tr>
<tr>
<td>4</td>
<td>47.43</td>
</tr>
<tr>
<td>5</td>
<td>46.94</td>
</tr>
<tr>
<td>6</td>
<td>46.43</td>
</tr>
<tr>
<td>7</td>
<td>47.11</td>
</tr>
<tr>
<td>8</td>
<td>46.56</td>
</tr>
<tr>
<td>9</td>
<td>46.73</td>
</tr>
<tr>
<td>10</td>
<td>46.80</td>
</tr>
<tr>
<td>( \bar{y} )</td>
<td>46.86</td>
</tr>
<tr>
<td>( s )</td>
<td>0.46</td>
</tr>
</tbody>
</table>
Example – Steady State Models

Point estimator
\[ \bar{Y}.(T_0 + T_E, T_0) = 46.86 \]

Standard error
\[ s.e.(\bar{Y}.T_0 + T_E, T_0)) = \frac{S}{\sqrt{R}} = 0.15 \]

\[ \alpha = 0.05, t_{0.025,9} = 2.26, 95\% \text{ CI for } w \]
\[ 46.86 - 2.26(0.15) \leq w \leq 46.86 + 2.26(0.15) \]

\[ 46.52 \leq w \leq 47.20 \]

The long-run mean cycle time is between 46.52 and 47.2 hrs.

We are assuming any significant initialization bias has been removed

---

Steady State Models – Warm Up and Run Length

- **Warm-up and run length times?**
  - Most practical idea: preliminary runs, plots
  - Simply “eyeball” them
  - Be careful about variability — make multiple replications, superimpose plots
  - Make R replications, initializing and deleting uniformly across them

- **Possibility – different Warm-up Periods for different output processes**
  - To be conservative, take the max
  - Must specify a single Warm-up Period for the whole model
Initialization Bias – Steady State Simulations

- FastChip wafer fabrication problem
- \( R = 10 \) independent replications
- 250 cycle times collected

\[ Y_{r,1}, Y_{r,2}, \ldots, Y_{r,250} \]

\[ \bar{Y}_j = \frac{1}{R} \sum_{r=1}^{R} Y_{rj} \]

Initialization Bias – Steady State Simulations

- Plot output from replication 1
- \( Y_{1,1}, Y_{1,2}, \ldots, Y_{1,250} \)
- Cumulative average

\[ \bar{Y}_{1\ell} = \frac{1}{\ell} \sum_{j=1}^{\ell} Y_{1j} \]