Proofs for Homework 1

Here are some examples of proofs for the homework problem 1.14.

*Sipser 1.14 a)* The set of Regular Languages is closed under compliment.

*Proof.* This proof is by construction.

Let $B$ be a regular language that is accepted by the DFA $M = (Q, \Sigma, \delta, q_0, F)$. Construct $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$. The accepting states, $F'$, are the complement of the accepting states of the original machine. Since $M'$ uses the same start state and transition function as $M$, $M'$ is a DFA.

It remains to be shown that $M'$ accepts all and only the words that are in the complement of $B$. Let the word $w \in B$ with terminal state $q \in Q$. With $w \notin B$, then $q \notin F$ and, by construction, $q \in F'$. Therefore $M'$ accepts all of the words in the complement of $B$. For the converse, consider a word $w$ that is accepted by $M'$ terminating in state $q \in F'$. By construction, $q \notin F$ and, therefore, $w \notin B$. This shows that $M'$ accepts only words not in language $B$.

We have shown that $M'$ exists, is a DFA, and accepts exactly the words in the complement of $B$. We can conclude that the complement of a regular language has an accepting DFA and is therefore regular.

*Super Structured Sipser 1.14 a)* The regular languages are closed under complement.

*Proof.* Given any DFA $M = (Q, \Sigma, \delta, q_0, F)$ for regular language $B$, we show how to construct a DFA for the language $\overline{B}$, showing that the complement of a regular language is also regular and that the regular languages are closed under complement.
1. Let $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$. Define construction for $M'$
2. $M'$ is a DFA $M$ and $M'$ have same $Q$ and $\delta$.
3. Given any string $w = w_1 w_2 \ldots w_n$ in $\Sigma^*$ there is a series of states $s_0, s_1, \ldots, s_n$ in $Q$ that obey transition function $\delta$ on $w$ with $s_0 = q_0$. Definition of computation in DFA
4. $s_n$ is in $F$ if and only if $s_n$ is not in $Q \setminus F$ Definition of set complement
5. $M$ accepts $w$ if and only if $M'$ rejects $w$ Definition of acceptance for DFA
6. The language accepted by $M'$ is the complement of the language accepted by $M$. Definition complement of a language.

**Unstructured** *Sipser 1.14 a*) The regular languages are closed under complement.

*Proof.* Given any DFA $M = (Q, \Sigma, \delta, q_0, F)$ for regular language $B$, we show how to construct a DFA for the language $\overline{B}$, showing that the complement of a regular language is also regular and that the regular languages are closed under complement.

Let $M' = (Q, \Sigma, \delta, q_0, Q - F)$. Because $M$ and $M'$ share the same states, transition function and start state, they are both DFAs and any string in the language will have identical computation paths through both machines. The only difference between the computations is whether the final state is an accept state. By definition of set complement, the last state will be in $F$ if and only if it is not in $Q \setminus F$. These are $M$ and $M'$’s respective accept states, so $M$ accepts a string $w$ if and only if $M'$ rejects it. Therefore the language computed by $M'$ is the complement of the language $B$. 

\[ \square \]