Main topics of the week:

- Nonregular Languages: Pumping lemma and examples

Pumping Lemma Examples

Consider the language $B = \{0^n1^n \mid n \geq 0\}$ that we intuitively feel is not regular. Suppose that $B$ is regular. Then by the pumping lemma, there is a pumping length $p$. Consider the string $0^p1^p$, and let $x$, $y$, and $z$ be the substrings guaranteed by the pumping lemma. It is certainly true that $y$ must consist of just zeroes, or just ones, or both. If $y$ consists of just zeroes, then the string $xyyz$ will have more zeroes than ones since the $y$ adds only more zeroes, and $x$ and $z$ are fixed. Likewise, if $y$ consists of just ones, then the same reasoning shows that $xyyz$ has more ones than zeroes. So we know that $y$ cannot be all zeroes or all ones. So $y$ must have some of each. However, if we again look at the string $xyyz$, we would see that the zeroes of the second $y$ occurrence would occur after the ones in the first $y$ occurrence, again a contradiction. Since all of the possibilities for $y$ lead to a contradiction, we conclude that there can be no pumping length, hence $B$ is not a regular language.

This proof can be simplified by using condition 3 from the pumping lemma: that $|xy| \leq p$. Using this, we can assert that $y$ must consist entirely of zeroes and thus not have to argue all the separate cases.

The art of using the pumping lemma is being clever about the choice of the string that produces a contradiction. Above it was pretty easy since any string of length $p$ would have worked, but that is not always the case.

For another example, let $P$ be the language of all palindromes, i.e., strings that are the same when read in the reverse direction. If $P$ is regular, then we have a pumping length $p$. Consider the string $a^pba^p$. Certainly this is a palindrome. In our decomposition $xyz$, we are guaranteed that the length of $xy$ does not exceed $p$. Thus, $xy$ must consist just of a’s and in particular this will be true of $y$, say $y=a^k$, where $k \geq 1$. But then $xz$ would have the form $a^{p-k}ba^p$, and this is obviously not a palindrome, so $P$ is not regular.

Here’s another example that uses a little more reasoning about lengths. Let $T = \{0^k \mid k = 2^n \text{ for some } n \geq 0\}$. That is, $T$ is the set of all strings of zeroes whose length is a power of 2. If it is regular, then the pumping lemma applies. Let $s$ be a string of length $m \geq p$, and $s=xyz$. We know that $y = 0^n$ for some $n$ and $m = 2^k$ for some $k \geq 0$. Note that $xz$ will have length $2^k - n$, which must also be a power of 2, say $2^k - n = 2^j$ for some $j < k$. Similarly, if we look at $xyyz$, we see that $2^k + n = 2^j$ for some $j > i$. Adding these together gives $2^{k+1} = 2^j + 2^j = 2^j(1 + 2^{j-i})$. Dividing out the $2^j$ leaves us with the left hand side a power of two, hence even, and the right hand side as 1 plus a power of two (power at least one), which is odd. Thus $T$ is not regular.