corresponding operation on generating functions is the simple operation of multiplication.

To illustrate the use of generating functions, we now discuss the challenging problem of counting the number of binary trees with \( n \) nodes. The convolution formula plays a prominent role in the solution to this problem.

Recall from Chapter 2 that a binary tree \( T_n \) of \( n \) nodes is empty if \( n = 0 \). If \( n > 0 \), it is a triple \((T_{n-1}, T_{n-1}, T_{n-2})\), where \( r \) is a distinguished node called the root of \( T_n \). \( T_r \) is a binary tree of \( l \) nodes for some \( l = 0, 1, \ldots, n-1 \), called the left subtree of \( T_r \), and \( T_{n-1} \) is a binary tree of \( n-l-1 \) nodes, called the right subtree of \( T_r \).

What is the number \( x_n \) of distinct binary trees \( T_n \) of \( n \) nodes? From the definition it follows that \( x_0 = 1 \), and obviously \( x_1 = 1, x_2 = 2, x_3 = 5 \). From the definition of \( T_r \), we deduce the recurrence relation

\[
x_n = \sum_{k=0}^{n-1} x_k x_{n-1-k}.
\]

(3.23)

This recurrence relation is rather difficult to solve by the techniques of Section 3.2. We would easily discover that \( x_n \) grows faster than a polynomial in \( n \) but that an exponential function \( e^{cn} \) grows too fast. It would then be a matter of luck to find a functional form that satisfies the recurrence relation.

We notice, however, that the right-hand side of (3.23) is a convolution and hence consider the generating function

\[
X(s) = \sum_{n=0}^{\infty} x_n s^n.
\]

Knowing that convolution of sequences corresponds to multiplication of generating functions, we write

\[
X(s) - \left( \sum_{n=0}^{\infty} x_n s^n \right)^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} x_k x_{n-k} \right) s^n.
\]

Since replacing \( n \) by \( k + 1 \) in (3.23) yields

\[
x_{k+1} = \sum_{i=0}^{k} x_i x_{k-i}
\]

we can simplify the right-hand side to obtain the equation

\[
X(s) - \frac{1}{s} \sum_{k=1}^{\infty} x_k s^{k+1} = \frac{1}{s} \left[ X(s) - x_0 \right]
\]

for the generating function. Using \( x_0 = 1 \), we obtain the solution

\[
X(s) = \frac{1}{2s} \left( 1 + \sqrt{1 - 4s} \right).
\]

(3.24)

The binomial expansion of \( \sqrt{1 - 4s} \) gives us

\[
(1 - 4s)^{1/2} = \sum_{k=0}^{\infty} \left( \binom{1/2}{k} \right) (-4s)^k
\]

\[
= 1 - \frac{1}{2} 4s + \frac{1}{2 \cdot 3!} (-4s)^2 - \frac{1}{2 \cdot 3 \cdot 4!} (-4s)^3 + \cdots
\]

The coefficient of \( s^k \), \( k \geq 1 \), in this series can be written as

\[
\binom{1/2}{k} = \frac{1}{k!} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{2^k (k - 1) \cdot 3 \cdot 5 \cdots (2k - 3)} = \frac{2}{k} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 3) (-1)^{k-1}}{(k - 1)(k-2) \cdots 2}.
\]

so that

\[
(1 - 4s)^{1/2} = 1 - \sum_{k=1}^{\infty} \frac{2}{k!} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{k - 1} s^k.
\]

(3.25)

Since the result of substituting (3.25) into (3.24) must be a series in nonnegative powers of \( s \), the plus solution \( X(s) = (1/2s)(1 + \sqrt{1 - 4s}) \) in (3.24) is extraneous. Substituting (3.25) into the solution

\[
X(s) = \frac{1}{2s} (1 - \sqrt{1 - 4s})
\]

yields

\[
X(s) = \sum_{n=0}^{\infty} \frac{1}{n + 1} \left( \frac{2n}{n}\right)^n.
\]

Therefore the number of binary trees of \( n \) nodes is

\[
x_n = \frac{1}{n + 1} \left( \frac{2n}{n}\right)^n
\]

for \( n \geq 0 \).

(3.26)

It is usual to find a simple closed-form expression for a nontrivial combinatorial sequence. Closed-form expressions are elegant and occasionally useful, but they do not always provide the answer in the most convenient form. For example, to determine the growth rate of \( x_n \) as a function of \( n \), we rewrite (3.26) as

\[
x_n = \frac{1}{n + 1} \left( \frac{2n}{n}\right)^n
\]

and use Stirling's formula, equation (3.12), to obtain

\[
x_n = \frac{4^n}{n^{n/2} e^{n/2}} \quad \text{as} \quad n \to \infty.
\]

(3.27)