Ch5  Linear Systems

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Two-dimensional Linear System

- Two-dimensional linear system
  - $x = ax + by$
  - $y = cx + dy$
  - If $x_1$ and $x_2$ are solutions, so is any linear combination $c_1x_1 + c_2x_2$
  - $x=0$ is always a fixed point

\[
\begin{align*}
  x &= A x \\
  A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x \\ y \end{pmatrix}
\end{align*}
\]
Example 5.1.1

• Example 5.1.1 Simple harmonic oscillator

\[ m \cdot \ddot{x} + k \cdot x = 0 \]

\[ x = \dot{v} \]

\[ \dot{v} = - \frac{k}{m} \cdot x \]
Example 5.1.1

\[ x = v \]

\[ v = -\omega^2 x \]

Vector field: a vector \( (x, v) = (v, -\omega x) \)

at each point \((x, v)\)
Definition

- Saddle point
- Stable manifold
- Unstable manifold
Stability Language

- Attracting
  \[ x(t) \rightarrow x^* \text{ as } t \rightarrow \infty \]
- Global attracting
  \[ x^* \text{ attracts all trajectories} \]
- Liapunov stable
  - All trajectories that start sufficiently close to
    Remain close to it
- Neutrally stable: Liapunov stable but not \( x^* \) attracting
- Asymptotically stable (stable)
  Both Liapunov stable and attracting
Classification of Linear Systems

• Straight-line trajectories: a trajectory starting on one of the coordinate axes stayed on that axis forever, and exhibited simple exponential growth or decay along it.

• General case \[ x(t) = e^{\lambda t} v \]

• \( V \) is an eigenvector and \( \lambda \) is the eigenvalue \[ A v = \lambda v \]
Characteristic Equation

- Characteristic equation \( A\nu = \lambda \nu \)
  - To find straight-line trajectories
- \( \text{det}(A-\lambda I)=0 \)
- For
  \[
  A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{det}\left( \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right) = 0
  \]
- \( \tau = \text{trace}(A) = a + d \)
- \( \Delta = \text{det}(A) = ad - bc \)
- \( \lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \quad \lambda_2 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \)
- \( x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \)  
  initial condition: \( x_0 = c_1 v_1 + c_2 v_2 \)
Characteristic of the Phase Plane

- $\lambda_1 > \lambda_2 > 0$ unstable nodes
- $\lambda_1 = \lambda_2$
  - All trajectories are straight lines when there are two eigenvector
  - Degenerate node when there is only one eigenvector
- $\lambda_1 > 0 > \lambda_2$
  - $v_1$ is the unstable manifold
  - $v_2$ is the stable manifold
  - Trajectory approaches $v_1$ as $t \to \infty$ and $v_2$ as $t \to -\infty$
- $0 > \lambda_1 > \lambda_2$
  - Both $v_1$ and $v_2$ are stable manifold
  - Trajectories approach $v_2$ as $t \to \infty$ and $v_1$ as $t \to -\infty$
Complex Eigenvalues

- Either a center or a spiral

\[ e^{(\alpha \pm iw)t} \]

\[ e^{iwt} = \cos(wt) + isin(wt) \]

- If \( \alpha > 0 \): growing oscillations (unstable)
- If \( \alpha < 0 \): decaying oscillations (stable)
- If \( \alpha = 0 \): ellipse
CH6 Phase Plane
Phase Plane

- Nonlinear systems
  - Typically no hope of finding the trajectories analytically
- Qualitative behavior
  - Fixed points
  - Closed orbits
  - Arrangement of trajectories near the fixed points and closed orbits
  - Stability and instability of the fixed points and closed orbits
Existence, Uniqueness, and Topological Consequences

- Existence and uniqueness of solutions are guaranteed if $f(x)$ is continuously differentiable
  - Different trajectories never intersect
- Poincare-bendixson theorem: if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit.
Fixed Points and Linearization

- Approximate the phase portrait near a fixed point
  - If the fixed point for the linearized system is not one of the borderline cases
- Using Taylor series expansion

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix} + \text{quadratic terms}
\]

\[
f(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{f''(\alpha)}{2!}(x - \alpha)^2 + \frac{f'''(\alpha)}{3!}(x - \alpha)^3 + \ldots + \frac{f^{(n)}(\alpha)}{n!}(x - \alpha)^n + \ldots
\]
Example 6.3.1

- Find all the fixed points of the system
  - \( x = -x + x^3 \)
  - \( y = -2y \)

- Fixed points \((0,0),(1,0),(-1,0)\)
- \((0,0)\) is stable, both \((1,0)\) and \((-1,0)\) are saddle points
6.4 Rabbits verse Sheep

The population growth depends on the food supply.

- Each species grow to its carrying capacity in the absence of the other
- When the two meet, usually sheep nudges the rabbit aside and starts nibbling, though some times the rabbit gets to eat
model

\[
\begin{align*}
& x = x (3 - x - 2y) \\
& y = y (2 - x - y)
\end{align*}
\]

- Four fixed point (0,0), (0,2), (3,0) and (1,1)

\[
A = \begin{pmatrix}
3 - 2x - 2y & -2x \\
-2y & 2 - x - 2y
\end{pmatrix}
\]

- (0,0) is unstable, (0,2), (3,0) is stable
- (1,1) is a saddle point
- Basin of attraction and basin boundary
6.5 **Conservative System**

\[ m x + \frac{dV}{dx} = 0 \]

- \( V(x) \) denote the potential energy (energy is conserved)

\[ m x x + \frac{dV}{dx} x = 0 \Rightarrow \frac{d}{dt} \left[ \frac{1}{2} m x^2 + v(x) \right] = 0 \]

\[ E = \frac{1}{2} m x^2 + v(x) \text{ is constant as a function of time} \]
Example 6.5.2

• Show that a conservative system cannot have any attracting fixed point
  • Suppose $x^*$ were an attracting fixed point
    • All points in its basin of attraction would have to be at the same energy $E(x^*)$
    • This contradicts our definition of a conservative system, in which we required that $E(x)$ be nonconstant on all open sets
Example 6.5.2

• Consider a particle of mass $m=1$ moving in a double-well potential

$$\nu(x) = -\frac{1}{2} x^2 + \frac{1}{4} x^4$$

• Find and classify all the equilibrium points for the system
Example 6.5.2

\[
\begin{align*}
\cdots \\
x &= x - x^3 \\
\cdots
\end{align*}
\]

- Fixed point \((0,0), (-1,0), \text{and} (1,0)\)
- \((0,0)\) is a saddle point,
- \((-1,0), (1,0) \) are centers
Energy Surface

- Trajectories must maintain a constant height $E$, so they would run around the surface, not down it.
- Centers occur at the local minima of the energy function
  - Neutral stable and oscillations to occur at the bottom
Reversible Systems

- Time-reversal symmetry
  - Their dynamics look the same whether time runs forward or backward
  - E.x. Undamped pendulum
- Any system of form \( m \ddot{x} = F(x) \) is symmetric under time reversal
  \[
  \begin{align*}
  x &= y \\
  y &= \frac{1}{m} F(x)
  \end{align*}
  \]
- Every trajectory has a twin: they differ only by time-reversal and a reflection in the x-axis
Reversible System

• Generally, for

\[ x = f(x, y) \]
\[ y = g(x, y) \]

• Reversible system: Any second-order system that is invariant under t->-t and y->-y.
Theorem 6.6.1

• Suppose the origin \( x^* = 0 \) is a linear center for the continuously differentiable system and suppose that the system is reversible. Then sufficiently close to the origin, all trajectories are closed curves.
Example 6.6.1

• Show that

\[ \begin{align*}
  \dot{x} &= y - y^3 \\
  \dot{y} &= -x - y^2
\end{align*} \]

Has a nonlinear center at the origin, and plot the phase portrait

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \tau = 0, \Delta > 0 \]

The origin is a center
Example 6.6.2

• Using reversibility arguments alone, show that the system

  \[
  \begin{align*}
  x &= y \\
  y &= x - x^2
  \end{align*}
  \]

  has a homoclinic orbits in the half-plane \( x \geq 0 \)

• General definition of reversibility:
  any mapping \( R(x) \) of the phase space to itself that satisfies \( r^2(x) = x \)
Example 6.6.3

• Show that

\[ x = 2 \cos x - \cos y \]

\[ y = 2 \cos y - \cos x \]

is reversible, but not conservative

• It has an attracting fixed point -> it is not conservative