Discussion of Problem 2, Homework 1

Problem. Solve the timing recurrence

\[ T(n) = T(\lceil \alpha n \rceil) + T(\lceil \beta n \rceil) + n \]

where \( \alpha, \beta < 1 < \alpha + \beta \).

Remark. Our solution will be somewhat long-winded in order to bring out the motivation. A straightforward proof would be far more concise— but then the constants appear to be pulled out of a hat.

Solution: Let \( \gamma \) satisfy

\[ 1 = \alpha^\gamma + \beta^\gamma. \]

Then \( \gamma > 1 \) and is uniquely determined since the function \( f(x) = \alpha^x + \beta^x \) is strictly decreasing. We show that

\[ T(n) = \Theta(n^\gamma) \]

We first show \( T(n) = \Omega(n^\gamma) \), that is, for some positive constants \( c, n_0 \), \( T(n) \geq cn^\gamma \) for \( n \geq n_0 \).

If we are not concerned about getting an optimal \( c \), we can just take \( n_0 = 1 \). Let \( m_0 \) be the minimum integer such that

\[
\text{for } m \geq m_0, \ [\alpha m] < m \text{ and } [\beta m] < m
\]

(we need these facts to carry out the induction). Now choose \( c \) such that \( T(m) \geq cm^\gamma \) for \( 1 \leq m < m_0 \). We show that \( T(n) \geq cn^\gamma \) for all \( n \).

For the inductive step, let \( n \geq m_0 \) and assume that \( T(m) \geq cm^\gamma \) for \( m < n \). Then

\[
T(n) = T(\lceil \alpha n \rceil) + T(\lceil \beta n \rceil) + n \\
\geq c[\alpha n]^\gamma + c[\beta n]^\gamma + n \\
\geq c(\alpha n)^\gamma + c(\beta n)^\gamma + n \\
= cn^\gamma + n \\
> cn^\gamma
\]

We next show \( T(n) = O(n^\gamma) \), that is, for some positive constants \( k, n_1 \), \( T(n) \leq kn^\gamma \) for \( n \geq n_1 \). Two obstructions arise if we just try to turn the inequalities around in the \( \Omega \) argument: the extra “\( n \)” and the fact that \( [\alpha n] \) remains \( \geq \alpha n \), while we need a reverse inequality. We can account for the “\( n \)” term in the induction by adding a \( k_2 n \) term to the purported upper bound (with \( k_2 \) picked to make the induction go through). But to get a reverse inequality for the \( \lceil \cdot \rceil \) terms, we cannot just drop the brackets this time; the best we can say is \( \lceil x \rceil < x + 1 \). Since we will need \( (\alpha n)^\gamma \) and \( (\beta n)^\gamma \) to turn up if we are to apply the relation defining \( \gamma \), we observe

\[
[\alpha n]^\gamma < (\alpha n + 1)^\gamma = (\alpha n)^\gamma(1 + \frac{1}{\alpha n})^\gamma = (\alpha n)^\gamma(1 + \text{something})
\]

where, intuitively, the “something” is tiny. We need to make that more precise but still not messy. Basically, we want a neat upper bound for \((1 + t)^\gamma\) for “small” \( t \). For example,
a one-term Maclaurin expansion with the appropriate remainder term will suffice: for any \( t \geq 0 \),
\[
(1 + t)^\gamma = 1 + \gamma (1 + t_0)^{\gamma - 1} t
\]
where \( 0 \leq t_0 \leq t \). If we simply restrict to \( t \leq 1 \), then for any such \( t \),
\[
(1 + t)^\gamma \leq 1 + \delta t
\]
where \( \delta = \gamma 2^{\gamma - 1} \). Thus, as long as \( \alpha n \geq 1 \),
\[
[\alpha n]^\gamma < (\alpha n + 1)^\gamma = (\alpha n)^\gamma (1 + \frac{1}{\alpha n})^\gamma \leq (\alpha n)^\gamma + \delta (\alpha n)^{\gamma - 1}.
\]
Putting it all together, we need a stronger induction hypothesis of the form
\[
T(n) \leq kn^\gamma - k_1 n^{\gamma - 1} - k_2 n.
\]  
(1)

We will decide on positive \( k, k_1, k_2 \) when we see what is needed for the induction (the reason for negative coefficients \( -k_1, -k_2 \) will also be clear at that point). First of all, \( n \) must be large enough to ensure \( \alpha n \geq 1, \beta n \geq 1 \). However, we will see that the starting point has to satisfy a stronger condition. So let us just say for now that \( m_1 \) is a mysterious integer that we will fix later and then \( m_2 \) is the minimum integer such that, for \( n \geq m_2 \), we have \( \alpha n \geq m_1, \beta n \geq m_1 \) as well as the condition needed to apply the induction hypothesis, namely, \( n > [\alpha n], n > [\beta n] \).

It suffices to prove (1) for \( n \geq m_1 \) and, furthermore, we will start the inductive reasoning at \( n = m_2 \). Thus, \( k, k_1, k_2 \) will also have to be chosen so that (1) automatically holds when \( m_1 \leq n < m_2 \).

For the inductive part, we take \( n \geq m_2 \) and assume that \( T(m) \leq km^\gamma - k_1 m^{\gamma - 1} - k_2 m \) for \( m_1 \leq m < n \). Then
\[
T(n) = T([\alpha n]) + T([\beta n]) + n
\]
\[
\leq (k[\alpha n]^\gamma - k_1[\alpha n]^{\gamma - 1} - k_2[\alpha n])
+ (k[\beta n]^\gamma - k_1[\beta n]^{\gamma - 1} - k_2[\beta n])
+ n
\]
\[
\leq k((\alpha n)^\gamma + \delta(\alpha n)^{\gamma - 1}) - k_1[\alpha n]^{\gamma - 1} - k_2[\alpha n])
+ k((\beta n)^\gamma + \delta(\beta n)^{\gamma - 1}) - k_1[\beta n]^{\gamma - 1} - k_2[\beta n])
+ n
\]
\[
\leq k((\alpha n)^\gamma + (\beta n)^\gamma)
+ (\delta k - k_1)((\alpha n)^{\gamma - 1} + (\beta n)^{\gamma - 1})
+ n - k_2(\alpha n + \beta n)
\](Note that we have also used \(-[\alpha n] < -\alpha n \) and \(-[\beta n] < -\beta n \)) Now we see which \( k_1, k_2 \) would ensure the last expression is \( \leq kn^\gamma - k_1 n^{\gamma - 1} - k_2 n \). We would like
\[
(\delta k - k_1)(\alpha^{\gamma - 1} + (\beta n)^{\gamma - 1}) \leq -k_1
\]
or
\[
\delta k(\alpha^{\gamma - 1} + (\beta n)^{\gamma - 1}) \leq k_1(\alpha^{\gamma - 1} + (\beta n)^{\gamma - 1} - 1)
\]
Recall that \( (\alpha^{\gamma-1} + \beta^{\gamma-1}) > 1 \) so we just take \( k_1 = k \varepsilon \) where

\[
\varepsilon = \frac{\delta (\alpha^{\gamma-1} + \beta^{\gamma-1})}{\alpha^{\gamma-1} + \beta^{\gamma-1} - 1}
\]

(though we have not yet decided on \( k \)). For \( k_2 \) we would like

\[ 1 - k_2 (\alpha + \beta) \leq -k_2 \]

so let us set

\[ k_2 = \frac{1}{\alpha + \beta - 1}. \]

Finally, for the induction to get its start, we need to choose \( k \) so that (1) holds for \( m_1 \leq n < m_2 \). Recall, however, that we did not yet specify \( m_1 \). The reason is that (1) has become

\[
T(n) \leq kn^{\gamma-1}(n - \varepsilon) - k_2 n
\]  

(2)

It would be impossible to find \( k \) enforcing (2) for any \( n \leq \varepsilon \). So choose \( m_1 = 1 + \lceil \varepsilon \rceil \). With \( m_1 \) now fixed, \( m_2 \) is fixed as above. We need only choose \( k \) large enough to guarantee

\[
T(n) \leq kn^{\gamma-1}(n - \varepsilon) - k_2 n, \text{ for } m_1 \leq n < m_2
\]

We have at last that (1) holds for these choices of \( k, k_1, k_2 \) when \( n \geq m_1 \). From which we can simply extract

\[
\text{for } n \geq m_1, \quad T(n) \leq kn^\gamma
\]