Take Home Final Problems

1. In considering “shortest paths” in edge-weighted graphs, it is sometimes desirable to consider the cost of a path to be the maximum of the weights on its edges (as opposed to the sum of the edge weights). With this notion of path-weight

   (a) Describe an algorithm for the single-source shortest-path problem.

   (b) Describe an algorithm for the all-pairs shortest-path problem

2. Consider “Lucky Medians”, Problem E on the 1997 UO Programming contest:
   http://www.cs.uoregon.edu/resources/progteam/dept_contest/
Assume that the array has odd length \( n \) and the entries are distinct.

   (a) Describe an efficient algorithm for this problem (in the course of which you are likely to have ideas about parts (b) and (c)).

   (b) Show that, if a median \( \mu \) can be determined from the given data, then one must also be able to determine the set \( L \) of elements less \( \mu \) and the set \( G \) of elements greater than \( \mu \).

   (c) Show that, if a median \( \mu \) can be determined from the given data, then the number of comparisons that did not compare some \( \lambda \in L \) with some \( \gamma \in G \) must be at least \( n - 1 \).

   (d) Show that any algorithm for finding the median in a set of \( n \) elements (\( n \) odd) requires at least \( \frac{3}{8}(n - 1) \) comparisons in the worst case.

   Hint: Show that an adversary can force you to have used at least \( \frac{n-1}{2} \) comparisons of the “\( \lambda \in L \) vs \( \gamma \in G \)” type before you even consider (his choice of) \( \mu \).

3. Let \( S \) be a finite set. Suppose \( g : 2^S \to 2^S \) satisfies

   (a) \( g(\emptyset) = \emptyset \).

   (b) \( A \subseteq g(A) \).

   (c) If \( A \subseteq B \) then \( g(A) \subseteq g(B) \).

   (d) \( g(g(A)) = g(A) \).

   (e) If \( b \in g(A \cup \{c\}) \setminus g(A) \), then \( c \in g(A \cup \{b\}) \).

Let \( \mathcal{I} = \{ A \in 2^S \mid C \subseteq A \subseteq g(C) \} \) implies \( A = C \).

Show that \((S,\mathcal{I})\) is a matroid.
4. Show that 2-Sat is in \( P \).

5. (a) Let \( G = (V, E) \) be a directed graph and \( k < |V| \) an integer. For each \( v \in V \), let \( N_k(v) \) denote the set of vertices of distance \( \leq k \) from \( V \). Describe an algorithm that determines \( N_k(v) \) for all \( v \in V \). For full credit, your algorithm should run in time \( \mathcal{O}(n^\alpha) \) with \( \alpha < 3 \).

(b) Suppose now that you need to determine a subset \( V' \subseteq V \) of minimum size such that \( \bigcup_{v \in V'} N_k(v) = V \). Is there likely to be an efficient algorithm for this problem? Explain (briefly if possible).

6. Show that the following problems are \textbf{NP-Complete}.

   (a) **Instance:** A graph \( G = (V, E) \) with \( |V| \) even.
   **Question:** Does \( G \) contain an independent set of size \( |V|/2 \)?

   (b) **Instance:** A directed graph \( G = (V, E) \).
   **Question:** Is there a subset \( W \) of \( V \) such that for \( v \in V \), there is an edge from \( v \) to some vertex in \( W \) iff \( v \notin W \)?

   \textit{Hint:} Reduction from 3SAT. It may be convenient to observe that, for such \( W \): if \( \{(u, v), (v, u)\} \) is a 2-cycle in \( G \), at most one of \( u, v \) can be in \( W \).