CIS 425—Assorted $\lambda$-Calculus Problems
(no due date)

These problems are presented here to give you a bit of experience in working with the $\lambda$-calculus. This document is not a homework assignment, and these do not need to be turned in. But have a shot at some or all of them. Either of us will be happy to go over your work with you.

### Problems

1. Give an example of a lambda-calculus expression in which a variable occurs both free and bound.

2. Give the normal forms for the following lambda terms, if one exists. If not, then state that no normal form exists. Show your reductions steps in all cases.
   
   (a) $(\lambda f.\lambda g.\lambda x.f(x))(\lambda x.\lambda y.x)(\lambda x.\lambda y.x)$
   
   (b) $\lambda x.\lambda y.x(\lambda z.y(\lambda w.zw))$
   
   (c) $(\lambda x.\lambda y.\lambda z.xyz)(\lambda x)y$
   
   (d) $(\lambda x.\lambda y.\lambda z.xyz)(\lambda x)y$
   
   (e) $(\lambda x.\lambda y.xxy)(\lambda x.xx)$
   
   (f) $(\lambda x.\lambda y.x(xy))(\lambda x.xx)$

3. $\lambda$-calculus is a universal computer language in the sense that each computable function can be represented by a $\lambda$-term. For example, we can represent numbers as follows:

   
   $0 \equiv \lambda xy.y$
   
   $1 \equiv \lambda xy.xy$
   
   $2 \equiv \lambda xy.x(xy)$
   
   $\vdots$
   
   $n \equiv \lambda xy.x^n y$

   (Read $x^n y$ as abbreviation for $\underbrace{x(x(\cdots (xy) \cdots )}_{\text{n times}}$). The above representation for numbers is referred to as **church numerals** in the literature. With the above representation, we can define the successor function $S$ as: $S \equiv \lambda xy.x(xy)$

4. Show that $S n = n + 1$, that is $S(\lambda x y.x^n y) = (\lambda x y.x^{n+1} y)$ for all $n \geq 0$.

5. Give the representation of the $+$ function as a $\lambda$-term. Suppose $+$ is defined by the lambda term $\lambda x y.P$. i.e., $+ \equiv \lambda x y.P$. Then it must be the case that

   $$(\lambda x y.P)(\lambda x y.x^n y)(\lambda x y.x^m y) = (\lambda x y.x^{n+m} y)$$

   In other words, $+ m n = m + n$. (Hint: Define the function $+$ using the successor function, i.e., $+ m n$ is obtained by applying the successor function $n$ times to $m$.) You should be convinced by now that you can represent everything in lambda calculus. You can check that the following makes sense. (It is not part of the exercise. It is only for satisfying your curiosity.)

   $\text{true} \equiv \lambda e t$

   $\text{false} \equiv \lambda e e$

   $\text{if} \equiv \lambda b e . b e$

   $\text{zero?} \equiv \lambda x.(\lambda y.\text{false})\text{true}$